# General Relativity and Cosmology

Winter term 2008/09

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Example sheet 13

1. **Observational Hubble law**

   Consider a Taylor expansion of the scale factor $a(t)$ about $a_0(t_0)$ and use this to express the redshift as

   $$z = H_0(t_0 - t) + \frac{1}{2}(2 + q_0)H_0^2(t_0 - t)^2 + \ldots , \tag{1}$$

   where you can identify the *decceleration parameter* $q = -a\ddot{a}/\dot{a}^2$ (the subscript $0$ denotes today). Invert this expression to get an expression for $t_0 - t$. By considering radial photon propagation ($d\theta = d\phi = 0$), find the expression for $r$

   $$r = a(t_0)^{-1}\left[(t_0 - t) + \frac{1}{2}H_0(t_0 - t)^2 + \ldots\right] \tag{2}$$

   Thus, use the *luminosity distance* $d_L = a_0\, r\,(1 + z)$ to find Hubble's law in terms of measurable quantities:

   $$H_0 d_L = z + \frac{1}{2}(1 - q_0)\, z^2 + \ldots \tag{3}$$

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Preparation questions for exam

1. **Killing verctors**

   Using Killing's equation, the formula for the commutator of two covariant derivatives,

   $$\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} = R^{\lambda}{}_{\sigma\rho\mu}\xi_{\lambda} \tag{4}$$

   and the cyclic sum rule for the Riemann tensor

   $$R^{\lambda}{}_{\sigma\rho\mu} + R^{\lambda}{}_{\rho\mu\sigma} + R^{\lambda}{}_{\mu\sigma\rho} = 0 \tag{5}$$

   show that

   $$\xi_{\mu;\rho;\sigma} = R^{\lambda}{}_{\sigma\rho\mu}\xi_{\lambda} \tag{6}$$

2. **Field theory in curved spacetime**

   Consider the action for a system of $n$ scalar fields

   $$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}G_{ab}\partial_\mu\phi^a\partial^\mu\phi^b - V(\phi^a)\right), \tag{7}$$

   where $G_{ab}(\phi^c)$ is a metric in the space of fields (but $R$ is the scalar curvature of spacetime as usual). Derive the equations of motion for the fields $\phi^a$ as well as the energy momentum tensor $T^{\mu\nu}$ for this action.

3. **Energy momentum tensor**

   (a) Given the energy momentum tensor for a perfect fluid

   $$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \tag{8}$$

   where $U^\mu = (1, 0, 0, 0)$. Compute $T^\mu_\nu$, $T$ and $\nabla_\mu T^{\mu 0}$ in a Friedmann-Robertson-Walker background.

   (b) Consider a spacetime whose Ricci tensor is given by $R_{\mu\nu} = 2\lambda g_{\mu\nu}$. Compute $R$ and show that the energy momentum tensor for such a geometry is given by

   $$T_{\mu\nu} = -\frac{3\lambda}{8\pi G}g_{\mu\nu} \tag{9}$$

4. **Black holes**

   Consider the Schwarzschild metric

   $$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2d\Omega^2 \tag{10}$$

   where $h = 1 - \frac{2M}{r}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

(a) How many Killing vectors has this metric? Discuss its spacetime structure.

(b) We know that for each Killing vector there exists a conserved quantity, or a constant of the motion for a free particle. If $K^\mu$ is a Killing vector, we know that

$$K_\mu \frac{dx^\mu}{d\lambda} = constant \tag{11}$$

where $\lambda$ is the affine parameter. What are the associated conserved quantities for the Schwarzschild metric?

(c) Consider the two Killing vectors associated to time translations and magnitude of the angular momentum

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0)$$

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1)$$

Use (11) to find expressions for the two associated conserved quantities of the above Killing vectors. What are this quantities?

(d) Besides the conserved quantities above, we always have another constant of the motion for geodesics. The geodesic equation (together with metric compatibility) implies that the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{12}$$

is a constant along the path and $\epsilon = \pm 1, 0$ (what do these values indicate?). Expand (12) and use the conserved quantities derived above to find an equation for $r(\lambda)$.

5. **Cosmology**

(a) In a FRW spacetime, the deceleration parameter $q$ is defined by

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2} \tag{13}$$

where dots denote derivatives with respect to cosmic time $t$. In a universe with a cosmological constant $\Lambda$, show that $q$ can be expressed in the matter era (with pressureless matter $p = 0$) as

$$q = \frac{1}{2}\Omega_M - \Omega_\Lambda \tag{14}$$

and in the radiation era ($p = \rho/3$) as

$$q = \Omega_M - \Omega_\Lambda \tag{15}$$

where $\Omega_M = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}}$ and $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ are the *density parameters* for matter and a cosmological constant.

(b) **Milne universe**. Consider the Friedmann equation for an empty universe, that is $\rho = 0$, but non-vanishing spatial curvature $k$. Describe the properties of such a universe in detail.