General Relativity and Cosmology

Winter term 2008/09

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Example sheet 3

Problems marked with an asterisk are for private study/homework.

1. **Tensors again(*)**

   (a) Suppose that in some coordinate system the components T_{ab} of a type[1] $(0, 2)$ tensor satisfy $T_{ab} = \delta_{ab}$. Show that this property is *not* coordinate invariant.

   (b) Verify that the relationship $T^{ab} = T^{ba}$, defining a symmetric tensor, is coordinate independent.

   (c) Show that if $S_{ab} = S_{ba}$ and $T^{ab} = -T^{ba}$ for all a, b, then $S_{ab}T^{ab} = 0$.

   (d) Show that any type $(0, 2)$ or type $(2, 0)$ tensor can be expressed as the sum of a symmetric and an antisymmetric tensor.

2. **Coordinate systems.**

   (a) Describe the curve given in spherical coordinates by

   $$r = a\,, \qquad \theta = t\,, \qquad \phi = 2t - \pi\,, \qquad 0 \leq t \leq \pi$$

   (where a is a positive constant) and find its length (you do not have to evaluate the integral!).

   (b) (*) Describe the curve given in cylindrical coordinates by

   $$\rho = a\,, \qquad \phi = t\,, \qquad z = t\,, \qquad -\pi \leq t \leq \pi$$

   and find its length.

   (c) Consider $\mathbf{R}^3$ with the flat Euclidean metric, and coordinates (x, y, z). Introduce spherical coordinates (r, θ, ϕ) related to (x, y, z) by

   $$x = r\sin\theta\cos\phi\,, \qquad y = r\sin\theta\sin\phi\,, \qquad z = r\cos\theta\,,$$

   so that the metric takes the form

   $$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,.$$

   i. A particle moves along a parametrised curve given by

   $$x(\lambda) = \cos\lambda\,, \qquad y(\lambda) = \sin\lambda\,, \qquad z(\lambda) = \lambda\,.$$

   Express the path of the curve in the (r, θ, ϕ) system.

   ii. Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

[1] That is, has only covariant components.

(d) (*) Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual cartesian coordinates (x, y, z) of Euclidean three-space by

$$x = \sinh\chi \sin\theta \cos\phi \,, \qquad y = \sinh\chi \sin\theta \sin\phi \,, \qquad z = \cosh\chi \cos\theta \,.$$

Restrict your attention to the plane $y = 0$ and answer the following questions.

i. What is the coordinate transformation matrix $\partial x^\mu / \partial x^{\nu'}$ relating (x, z) to (χ, θ) ?

ii. What does the line element ds^2 look like in prolate spheroidal coordinates?

3. **Geodesics.**

(a) Obtain the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda} = (e^{-1}\dot{e}) \frac{dx^\mu}{d\lambda}$$

directly from varying the action

$$S = -m \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

where $\dot{x} = dx/d\lambda$ and $me = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ as defined in the lectures.

(b) Imagine we have a *diagonal* metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^\lambda_{\mu\nu} = 0$$

$$\Gamma^\lambda_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$$

$$\Gamma^\lambda_{\mu\lambda} = \partial_\mu \left(\ln \sqrt{|g_{\lambda\lambda}|}\right)$$

$$\Gamma^\lambda_{\lambda\lambda} = \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|}\right)$$

In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are *not* summed over.

(c) (*) In Euclidean three-space, we can define paraboloidal coordinates (u, v, ϕ) via

$$x = uv\cos\phi \,, \qquad y = uv\sin\phi \,, \qquad z = \frac{1}{2}(u^2 - v^2) \,.$$

i. Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates $\partial x^\mu / \partial x^{\nu'}$ and the inverse transformation. Are there any singular points in the map?

ii. Find the metric and inverse metric in paraboloidal coordinates.

iii. Calculate the Christoffel symbols.

(d) Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \,.$$

Show that the lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).