# General Relativity and Cosmology

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Example sheet 5

1. **Parallel Transport**

   Consider a 2-sphere of radius $a$ with coordinates $(\theta, \phi)$ and metric

   $$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2\,.$$

   Take a unit vector $V^\mu$ whose direction makes an angle $\alpha$ east of south, and parallel-transport it once around a circle of constant latitude (that is $\theta = \theta_0 =$const.)

   (a) What are the components of the resulting vector?

   (b) Compute its magnitude and direction.

   (For this excersise, you can make use of the results obtained in example sheet 3, exercise 3(d).)

2. **Equivalence principle (EP) again: a rotating reference system**

   The EP implies that "fictitious" forces of accelerating coordinate systems are essentially in the same category as the "real" forces of gravity. Put another way, if the geodesic equation contains gravity in the $\Gamma^\mu_{\nu\sigma}$, it must also contain accelerations which have been built in by choice of coordinate system. In flat spacetime, these are the only sources of accelerations and these should be included in the $\Gamma^\mu_{\nu\sigma}$. In this exercise we consider an example of this.

   Start with a non rotating system $K$ with coordinates $(T, X, Y, Z)$ and Minkowski line element

   $$d\tau^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \qquad (1)$$

   Define new coordinates $(t, x, y, z)$ by

   $$\begin{aligned} T &\equiv t \\ X &\equiv x\cos(\omega t) - y\sin(\omega t) \\ X &\equiv x\cos(\omega t) - y\sin(\omega t) \\ Z &\equiv z \end{aligned} \qquad (2)$$

   (a) Draw a picture of the two coordinate systems. (Points given by $x, y, z$ constant, rotate with angular speed $\omega$ about the $Z$ axis of $K$. This defines the rotating system $K'$)

   (b) Find the form of the line element (the metric components) in the rotating coordinate system $K'$.

   (c) Compute all the geodesic equations using the proper time as the affine parameter.

   (d) Massage these equations and by introducing the mass $m$ of a free particle, show that these can be written in the form

   $$m\frac{d^2\vec{r}}{dt^2} = m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\,m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right) \qquad (3)$$

   where $\vec{r} \equiv (x, y, z)$ and $\vec{\omega} = (0, 0, \omega)$. Interpret this result.

3. **Covariant Derivative**

In this exercise we will obtain useful expression for the divergence $\nabla_\mu$ and the Laplacian $\nabla_\mu \nabla^\mu$.

(a) Show first that

$$\Gamma^\mu_{\nu\mu} = \frac{\partial(\ln\sqrt{-g})}{\partial x^\nu}.$$

(b) Show next that

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\, g^{\mu\lambda})}{\partial x^\mu}.$$

(c) Thus show that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\, V^\mu)}{\partial x^\mu}.$$

(d) Use these results to find a similar expression for the divergence of an an antisymmetric tensor $\nabla_\mu A^{\mu\nu}$.

(e) Use the above results to compute $\nabla_\mu \nabla^\mu \phi$, where $\phi$ is a scalar.

4. **Preparation for Schwarzschild**

The line element for a static spherically symmetric spacetime is given by

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4}$$

(a) For $A = B = 1$, show that this metric is that in eq. (1) in spherical coordinates.

(b) Compute all the non-vanishing Christoffel symbols (there are only 9!)