String Theory

Winter Term 2008/2009

Problem Sheet 4

Discussion: November 19, 14:00 in Hörsaal 118, AVZ

1. Consider a massless state in D-dimensional flat spacetime. What is the little group? Show that massless states transform in representations of $SO(D-2)$!

2. Derive the condition that $a = 1$ and $D = 26$ from the requirement that the Lorentz symmetry is not anomalous in light-cone gauge!

   The Lorentz algebra is generated by $J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}$, where $l^{\mu\nu}$ is the 'center-of-mass' angular momentum and $E^{\mu\nu}$ involves the oscillator states:

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\nu ,$$
$$E^{\mu\nu} = -\mathrm{i}\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) .$$

   The algebra to be fulfilled is

$$[J^{\mu\nu}, J^{\rho\sigma}] = (\mathrm{i}\eta^{\mu\rho}J^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) .$$

   (a) Argue that only the generators J^{+-} and J^{i-} can be anomalous!

   (b) Now consider only the commutator $[J^{i-}, J^{j-}]$. Show that the dangerous terms are of the form

$$\left[J^{i-}, J^{j-}\right] \sim \sum_{m=1}^{\infty} \Delta_m \left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right)$$

   with some coefficients Δ_m!

   (c) Show how to project out a given Δ_m by taking expectation values in an appropriate state! Evaluate that expectation value to find Δ_m!

3. The generalised ζ function (or Hurwitz ζ function) is defined for $\Re s > 1$ by

$$\zeta(s, b) = \sum_{n=0}^{\infty} (n+b)^{-s} .$$

   (A term where $n+b=$ is to be excluded from the sum.) It can be analytically continued (for $b > -1$) to a function holomorphic on the complete complex s-plane except $s = 1$ given by the series

$$\zeta(s, b) = \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \binom{n}{k} (b+k)^{1-s} .$$

   Clearly, for $b = 1$ the Hurwitz ζ function reduces to the Riemann ζ function.

   Calculate $\zeta(-1, b)$ and show that $\zeta(-1, 1) = -\frac{1}{12}$! (Hint: $\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^l$ vanishes for $0 \leq l < n$).

4. Following the last sheet, quantise the twisted closed string ($X^\mu(\sigma+\pi,\tau) = -X^\mu(\sigma,\tau)$) and the Neumann–Dirichlet open string ($X^\mu(0,\tau) = 0$ and $\partial_\sigma X^\mu(\sigma,\tau)\big|_{\sigma=\pi} = 0$)! Use the derived mode expansion and

   (a) find the commutation relations for the modes α_i from the canonical commutator

   $$[P^\mu(\sigma,\tau)\,, X^\nu\,(\sigma',\tau)] = -\mathrm{i}\eta^{\mu\nu}\delta(\sigma-\sigma')\ .$$

   (b) find the Virasoro generators L_0 (and $\widetilde{L}_0$)!

   (c) use ζ function regularisation to find the normal ordering constants a defined by $:\!L_0\!:= L_0 + a$!