String Theory

Winter Term 2008/2009

Problem Sheet 6

Discussion: December 3, 14:00 in Hörsaal 118, AVZ

1. For two matrices A and B of dimension $M \times N$ and $P \times Q$, the Kronecker product $A \otimes B$ is a $MP \times NQ$ matrix with elements

$$(A \otimes B)_{im,jn} = A_{ij} B_{mn} \,.$$

The indices run over $im = 11, 12, \ldots, NQ$ etc. For example, if A is a 2×2 matrix, the Kronecker product with B is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}$$

Clearly, it is linear and associative.

Check the following properties of the Kronecker product:

(a) Transpose and complex conjugation distribute over the Kronecker product, i.e.

$$(A \otimes B)^T = A^T \otimes B^T \,, \qquad (A \otimes B)^* = A^* \otimes B^* \,.$$

(b) If dimensions match, matrix multiplication factorises:

$$(A \otimes B)\,(C \otimes D) = (AC \otimes BD)$$

(c) If A and B are square matrices of dimensions $M \times M$ and $N \times N$, we have for the trace and determinant

$$\det(A \otimes B) = (\det A)^N \, (A \otimes B)^M \,, \qquad \operatorname{tr}(A \otimes B) = \operatorname{tr} A \cdot \operatorname{tr} B \,.$$

2. We can recursively define a particular representation of the Γ matrices for all dimensions in the following way: First we consider even dimensions. In $D = 2$, start with

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,.$$

Now assume the Γ matrices in $D - 2$ dimensions to be γ^μ. In D dimensions, define

$$\Gamma^\mu = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \gamma^\mu \,, \qquad \Gamma^{D-2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1} \,, \qquad \Gamma^{D-1} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \otimes \mathbb{1}$$

For odd dimensions, $D = 2k + 1$, we take the $2k$-dimensional Γ matrices and add Γ_* (see previous sheet).

(a) Show that this procedure defines a set of Γ matrices in any dimension! What about the ambiguity in α?

(b) Show that, in any dimensions, Γ^0 and all odd Γ^i for $i \geq 3$ are antisymmetric, while Γ^1 and the even Γ^i are symmetric! Conclude that Γ^3, Γ^5,…,Γ^9 are imaginary and the other ones are real.

3. Consider again a Majorana condition, i.e. a reality condition on a spinor of the form $\psi^* = B\psi$ (note the switch $B \to B^*$ relative to the previous sheet). As we saw, this requires

$$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *} \quad \text{and} \quad B^*B = 1\,.$$

Show this if you didn't already.

Consider even dimensions first. Use the Γ matrix representation defined in the previous problem. Let

$$B = \Gamma^3\Gamma^5\cdots\Gamma^{D-1}\,, \qquad\qquad B' = \Gamma_* B\,.$$

Show that

$$B\Gamma^\mu B^{-1} = -\left(-1\right)^{D/2}\Gamma^{\mu *}\,, \qquad\qquad B'\Gamma^\mu B'^{-1} = \left(-1\right)^{D/2}\Gamma^{\mu *}\,,$$

so both B and B' satisfy the first condition above. For which D do they also satisfy the second one?

Under which condition is the Majorana condition compatible with a chirality condition, $\Gamma_*\psi = \pm\psi$?

The definitions of B and B' also extend in $D+1$ dimensions. Do they both generate consistent Majorana conditions? In which dimensions?

4. Argue that the Γ matrices can be chosen such that Γ^0 to be anti-Hermitean and the Γ^i are Hermitean! (All commonly used representations have this property, or the opposite one if the convention for the metric is different.)

If the representation is chosen such, show that $\Gamma^0\Gamma^\mu\Gamma^0 = \Gamma^{\mu\dagger}$. (Clearly, the $\Gamma^{\mu\dagger}$ satisfy the same algebra as the Γ^μ. This equation implies that both representations are equivalent, and Γ^0 is called an intertwiner.)

Show further that $\bar\psi = \psi^\dagger\Gamma^0$ transforms in the conjugate representation of the Lorentz group,

$$\bar\psi \longrightarrow \bar\psi \exp\{-\mathrm{i}\omega_{\mu\nu}\Sigma^{\mu\nu}\}\ ,$$

so that $\bar\psi\psi$ is a Lorentz scalar.

In even dimensions, we can use the chiral projectors $P_\pm = \frac{1}{2}\left(1 \pm \Gamma_*\right)$ to form the chiral states $\psi_\pm = P_\pm\psi$. Show that a mass term $m\bar\psi\psi$ mixes ψ_+ and ψ_-, while the kinetic term $\mathrm{i}\bar\psi\Gamma^\mu\partial_\mu\psi$ does not!

5. Derive the massless particle spectrum of the closed string sector of type I string theory, i.e. start from type IIB and check the behaviour of the various fields under worldsheet parity reversal!