String Theory

Winter Term 2008/2009

Problem Sheet 7

Discussion: January 7, 14:00 in Hörsaal 118, AVZ

1. Show that

$$e^{\mathrm{i}\omega_{\rho\sigma}\Sigma^{\rho\sigma}}\Gamma^\mu e^{-\mathrm{i}\omega_{\rho\sigma}\Sigma^{\rho\sigma}} = \Gamma^\mu + 2\omega^\mu{}_\rho\Gamma^\rho + \mathcal{O}\big(\omega^2\big) \ .$$

   Deduce that $\bar{\psi}\Gamma^\mu\psi$ transforms as a vector, while $\bar{\psi}\Gamma^\mu\partial_\mu\psi$ is a scalar.

2. In the following, we will consider a five-dimensional theory where the fifth coordinate is curled up on a circle, $x^4 \equiv y \cong y + 2\pi R$. (The index M runs from 0 to 9, while the nine-dimensional indices are $\mu, \nu = 0, \dots, 8$.)

   (a) For starters, ignore gravity and take a free complex scalar field with the action

$$S = \int \mathrm{d}^5x \left(\left(\partial_M\phi\right)^\dagger \partial^M\phi + M^2\phi^\dagger\phi \right) \ .$$

   Show, using Fourier expansion, that the theory is equivalent to a four-dimensional theory with an infinite tower of fields ϕ_n with masses $m_n^2 \sim M^2 + n^2/R^2$! This is called the Kaluza–Klein tower.

   (b) Repeat the exercise for a ten-dimensional spinor Ψ! (The Lagrangean is $\mathscr{L} = \overline{\Psi}\Gamma^M\partial_M\Psi + M\overline{\Psi}\Psi$.)

   (c) Now turn to gravity. The metric in five dimensions is denoted by G_{MN}, the action is of the usual Einstein–Hilbert form

$$S = \frac{1}{16\pi G_{(5)}} \int \mathrm{d}^5x\sqrt{-G}\, R_{(5)}$$

   with the Ricci scalar $R_{(5)}$ constructed from G_{MN} and the five-dimensional Newton constant $G_{(5)}$.

   It is convenient to parametrise the metric and its inverse in the following form:

$$G_{MN} = \phi^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} , \qquad G^{MN} = \phi^{\frac{1}{3}} \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \frac{1}{\phi} + A_\mu A^\mu \end{pmatrix} .$$

   Now assume that $g_{\mu\nu}$, A_μ and ϕ do not depend on y. (Why could this assumption be justified?) Show that the theory reduces to four-dimensional general relativity coupled to a $U(1)$ gauge field and a scalar! What is the resulting four-dimensional Newton constant?

3. Now consider string theory on $\mathcal{M}^{1,8} \times S^1$ (where $\mathcal{M}^{1,8}$ is nine-dimensional Minkowski space). The radius of the S^1 is again denoted by R.

   We define an operator H which maps

$$H : \begin{cases} X_L^9 \longrightarrow \ \ X_L^9 \\ X_R^9 \longrightarrow -X_R^9 \end{cases} .$$

   (a) Recall the spectrum of the closed string, including Kaluza–Klein and winding modes. Is H a symmetry of the spectrum?

   (b) Find the spectrum for open strings with Neumann boundary conditions along X^9, i.e. $\partial_\sigma X^9\big|_{\sigma=0,\pi} = 0$! What is the action of H on the boundary conditions and the spectrum?

   (c) Repeat the exercise for Dirichlet boundary conditions, i.e. $X^9(\sigma = 0) = 0$, $X^9(\sigma = \pi) = l$!