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Theoretische Physik

Exercise 1
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Exercises on Theoretical Particle Physics

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–Class Exercises–

C 1.1 The Dirac equation

Using the operator substitutions $\vec{p} \to -\mathrm{i}\vec{\nabla}$, $E \to \mathrm{i}\,\partial_t$ it is possible to get the equations for quantum mechanics from the energy-momentum relations. From the non-relativistic equation $E = \frac{\vec{p}^2}{2m}$ one obtains the Schrödinger equation.

(a) Obtain the Klein–Gordon equation from the relativistic energy-momentum relation $E^2 = \vec{p}^2 + m^2$. Dirac's basic idea was to "factorize" the Klein–Gordon equation to obtain an equation which is first-order in the derivatives.

(b) Make the ansatz

$$H\psi = (\alpha_i\, p^i + \beta m)\psi\,. \tag{1}$$

Squaring the Hamilton operator eq. (1) and using $H^2\psi = E^2\psi$ should give the Klein–Gordon equation. Show that from this requirement it follows:

$$\beta^2 = \alpha_i^2 = \mathbb{1}, \qquad \{\beta, \alpha_i\} = \{\alpha_i, \alpha_j\} = 0, \quad i \neq j. \tag{2}$$

(c) Why are the α_i and the β not numbers? Why do they have to be hermitian ($A = A^\dagger$)? What does it imply?

(d) Define the Dirac matrices γ^μ, $\mu = 0 \ldots 3$ by

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\,\alpha_i, \quad i = 1, 2, 3. \tag{3}$$

Show that the Dirac equation $H\psi = E\psi$ can be written in the covariant form

$$(\mathrm{i}\,\gamma^\mu\partial_\mu - m)\,\psi = 0. \tag{4}$$

(e) Show that the gamma matrices fulfill the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}, \qquad \eta^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{5}$$

(f) Show the following relations:

$$\left(\gamma^0\right)^\dagger = \gamma^0, \qquad \left(\gamma^k\right)^\dagger = -\gamma^k \tag{6}$$

$$\left(\gamma^0\right)^2 = \mathbb{1}, \qquad \left(\gamma^k\right)^2 = -\,\mathbb{1}, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{7}$$

The lowest dimensional matrices satisfying the Clifford algebra eq. (5) are 4×4 matrices. The choice of the matrices is not unique. The following are two possible representations: The Weyl (or chiral) representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{8}$$

and the Dirac–Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{9}$$

Here σ_1, σ_2 and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

which satisfy the anti-commutation relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\, \mathbb{1}_{2\times2} \,. \tag{11}$$

(g) Verify that each set of matrices eqs. (8), (9) fulfills the Clifford algebra eq. (5).

C 1.2 Free solutions of the Dirac equation

Since H is represented by a 4×4 matrix, the ψ's are 4-component objects called *spinors*: $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$.

(a) Use the covariant form of the Dirac eq. (4) to show that for every ψ_α, $\alpha = 1 \ldots 4$:

$$\left(\Box + m^2\right) \psi_\alpha = 0 \,. \tag{12}$$

(b) For free particles it is possible to make the ansatz $\psi = u(\vec{p})\, \mathrm{e}^{-\mathrm{i}\, p\cdot x}$. Plugging it into eq. (1) and considering the Dirac–Pauli representation eq. (9) show that

$$Hu = \begin{pmatrix} m\, \mathbb{1}_{2\times2} & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\, \mathbb{1}_{2\times2} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}, \tag{13}$$

with u splitting into two 2-component spinors u_A and u_B.

(c) What are the energy eigenvalues for a particle at rest? Interpret the result.

(d) Now take $\vec{p} \neq 0$. We will label the solutions by an index s. You can find two solutions by choosing $u_A^{(s)} = \chi^{(s)}$ with $\chi^{(1)} = (0,1)^T$ and $\chi^{(2)} = (1,0)^T$. Which are the corresponding u_B? What can you say about the energy eigenvalues of these solutions? Proceed analogously for the remaining two solutions.

(e) It is convenient to choose the covariant normalization $\int \psi^\dagger \psi\, dV = 2|E|$. Use this to normalize the $u^{(s)}$'s.

From the solutions we see that there are always two solutions per eigenvalue. Such degeneracies are always due to additional symmetries.

(f) Show that the operator

$$\Sigma \cdot \hat{p} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix}, \qquad \hat{\vec{p}} = \frac{\vec{p}}{|\vec{p}|}, \tag{14}$$

corresponds to an observable, i. e. it commutes with H and P. The associated quantum number is called *helicity*.
Choose $\vec{p}$ along the z axis. What are the helicities of the $u^{(s)}$?

–Home Exercises–

H 1.1 γ-Matrix identities *1.5+5+3.5=10 points*

The following exercise is to be solved by only using the Clifford algebra of the γ-matrices and **not** a particular representation. For convenience we introduce the notation

$$\gamma^5 = \mathrm{i}\, \gamma^0 \gamma^1 \gamma^2 \gamma^3 \,. \tag{15}$$

(a) Show that

$$\left(\gamma^5\right)^\dagger = \gamma^5 \,, \qquad \left(\gamma^5\right)^2 = \mathbb{1} \,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0 \,. \tag{16}$$

(b) Prove the following trace theorems.

$$\mathrm{tr}\left(\gamma^\mu \gamma^\nu\right) = 4\eta^{\mu\nu} \tag{17}$$
$$\mathrm{tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \tag{18}$$
$$\mathrm{tr}\left(\gamma^{\mu_1} \dots \gamma^{\mu_n}\right) = 0 \,, \qquad \text{for } n \text{ odd} \tag{19}$$
$$\mathrm{tr}\, \gamma^5 = 0 \tag{20}$$
$$\mathrm{tr}\left(\gamma^\mu \gamma^\nu \gamma^5\right) = 0 \tag{21}$$
$$\mathrm{tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5\right) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma} \tag{22}$$

Hint: Use the cyclicity of the trace.

(c) Show the following contraction identities:

$$\gamma^\mu \gamma_\mu = 4 \cdot \mathbb{1} \tag{23}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{24}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho}\, \mathbb{1} \tag{25}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \tag{26}$$

H 1.2 The Lorentz group *1+2+3+3+1=10 points*

The Lorentz group is defined as the set of transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{27}$$

which leave the scalar product $\langle x, y\rangle = \eta_{\mu\nu}\, x^\mu y^\nu$ invariant.

1. Show that an element λ of the Lie algebra of the Lorentz group satisfies:

$$\lambda^T = -\eta\lambda\eta\,. \tag{28}$$

Hint: Reformulate the statement about the invariance of the scalar product in $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu$ *and write an element of the Lorentz group as* $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - i\lambda^\mu{}_\nu$.

2. Choose

$$(M^{\mu\nu})^\rho{}_\sigma = \mathrm{i}\,(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma) \tag{29}$$

as a basis for the Lie algebra. What do these matrices look like? Describe the form of the matrices in words. Verify the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\,(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho})\;. \tag{30}$$

3. We split the generators into two groups:

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}\,, \qquad K^i = M^{0i}\,. \tag{31}$$

The J's have only spatial indices, the K's have spatial and timelike indices. Verify the commutation relations

$$\left[J^i, J^j\right] = \mathrm{i}\,\epsilon^{ijk}J^k\,, \qquad \left[J^i, K^j\right] = \mathrm{i}\,\epsilon^{ijk}K^k\,, \qquad \left[K^i, K^j\right] = -\mathrm{i}\,\epsilon^{ijk}J^k\,, \tag{32}$$

and describe the meaning of each relation in words. What kind of transformations do the J's and K's correspond to?

4. The form of the commutation relations for the Lorentz algebra can still be simplified. Define

$$T^i_{\mathrm{L/R}} = \frac{1}{2}\left(J^i \pm \mathrm{i}\,K^i\right) \tag{33}$$

and verify the commutation relations

$$\left[T^i_{\mathrm{L}}, T^j_{\mathrm{L}}\right] = \mathrm{i}\,\epsilon^{ijk}\,T^k_{\mathrm{L}}\,, \qquad \left[T^i_{\mathrm{R}}, T^j_{\mathrm{R}}\right] = \mathrm{i}\,\epsilon^{ijk}\,T^k_{\mathrm{R}}\,, \qquad \left[T^i_{\mathrm{L}}, T^j_{\mathrm{R}}\right] = 0\,. \tag{34}$$

5. Classify the representations of the Lorentz algebra using what you learned about $\mathfrak{su}(2)$.

Conclusion: Every representation of the Lorentz algebra can be characterized by two non-negative integers or half-integers $(j_{\mathrm{L}},\, j_{\mathrm{R}})$.