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Exercise 2
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Exercises on Theoretical Particle Physics

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H 2.1 Representations of $\mathfrak{su}(2)$ *1+1.5+1+1+1+1.5+0.5+1.5+2=11 points*

A Lie algebra $\mathfrak{g}$ is a real vector space together with a smooth mapping $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

(i) The mapping is bilinear.

(ii) The mapping is skew-symmetric: $[a, b] = -[b, a]$ for $a, b \in \mathfrak{g}$.

(iii) It fulfills the Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for $a, b, c \in \mathfrak{g}$.

A *representation* ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping $\rho : \mathfrak{g} \to \mathrm{End}(V)$ which is an algebra homomorphism, i. e. $\rho([a, b]) = [\rho(a), \rho(b)]$. The dimension of V is called the dimension of the representation: $\dim(\rho) := \dim(V)$.

If there is a vector space $\{0\} \neq W \subsetneq V$ such that $\rho(W) \subset W$, the representation is called reducible and W is called the invariant subspace. If such W does not exist the representation is called *irreducible*; i. e. a representation is irreducible *if and only if* V is the only invariant subspace itself. In this exercise we will focus on the algebra $\mathfrak{su}(2)$.

(a) For $G \in \mathrm{SU}(2)$ we can write $G = \mathrm{e}^{\mathrm{i}g}$ with $g \in \mathfrak{su}(2)$. The group SU(2) is the set of all 2-dimensional unitary matrices with determinant 1. Show that the corresponding Lie algebra $\mathfrak{su}(2)$ is the set of all traceless hermitian matrices.
Hint: $\det A = \exp \mathrm{Tr} \log A$.

(b) Choose the basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

for the traceless hermitian matrices with the commutation relation $[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon^{ijk}\sigma^k$ and define

$$J_3 = \frac{1}{2}\sigma_3, \quad J_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2), \quad J_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2). \tag{2}$$

Verify the commutation relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3. \tag{3}$$

In the following let us consider **all irreducible, finite-dimensional representations** of $\mathfrak{su}(2)$ on a vector space V, $\rho(J_i) \in \mathrm{End}(V)$, $i = 3, +, -$. We will proceed stepwise in order to classify these representations and to find out which $\dim(V)$ are allowed.

(c) Since J_3 is diagonal, $\rho(J_3)$ can also be chosen to be diagonal. Therefore V can be decomposed into eigenspaces of $\rho(J_3)$,

$$V = \bigoplus V_\alpha \,, \tag{4}$$

where α labels the eigenvalues of $\rho(J_3)$, i. e.

$$(\rho(J_3))\, v = \alpha v, \quad v \in V_\alpha \,, \quad \alpha \in \mathbb{C} \,. \tag{5}$$

Show that $J_+(v) \in V_{\alpha+1}$ and $J_-(v) \in V_{\alpha-1}$.

Hint: For convenience use shorthand J_i for $\rho(J_i)$.

(d) Prove that all complex eigenvalues α which appear in the above decomposition differ from one another by 1.

Hint: Choose an arbitrary $\alpha_0 \in \mathbb{C}$ from the decomposition and prove that $\bigoplus_{k\in\mathbb{Z}} V_{\alpha_0+k} \subset V$ is indeed equal to V using the irreducibility of the representation.

(e) Argue that there is a $k \in \mathbb{N}$ for which $V_{\alpha_0+k} \neq \{0\}$ and $V_{\alpha_0+k+1} = \{0\}$. Define $n := \alpha_0 + k$. Note that up to now we only know that $n \in \mathbb{C}$. Draw a diagram. Write the vector spaces V_{n-2}, V_{n-1} and V_n in a row and indicate the action J_3, J_+ and J_- on these vector spaces by arrows. The eigenvalue n is called *highest weight* and a vector $v \in V_n$ is called *highest weight vector*. Why?

(f) Choose an arbitrary vector $v \in V_n$ (highest weight vector). Prove that the vectors v, J_-v, J_-^2v, ... span V.

Hint: Show that the vector space spanned by these vectors is invariant under the action of J_3, J_+ and J_- and use the irreducibility of the representation.

(g) Argue that all eigenspaces V_α are 1-dimensional.

(h) Prove that n is a non-negative integer or half integer and that

$$V = V_{-n} \otimes \ldots \otimes V_n \,. \tag{6}$$

Complement the diagram drawn in part (e). Which is the dimension of the representation?

Hint: The representation is finite dimensional, so there exists $m \in \mathbb{N}$ for which $J_-^{m-1}v \neq 0$ and $J_-^m v = 0$. Evaluate $J_+J_-^m v$.

(i) Consider the tensor product of a 2-dimensional and a 3-dimensional irreducible representations of $\mathfrak{su}(2)$:

$$V = V^{(2)} \otimes V^{(3)} \,. \tag{7}$$

Show that the resulting representation V is reducible and that it can be decomposed into a 2-dim. and a 4-dim. irreducible representation. Shorthand: $\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{4}$.

Hint: The action of a Lie algebra on the tensor product of two representations is given by: $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, i. e. the eigenvalue of J_3 on V is the sum of *the eigenvalues of J_3 on $V^{(2)}$ and $V^{(3)}$. Draw the diagrams of the eigenvalues(with multiplicities). Then use the fact that the eigenspaces of irreducible representations are 1-dimensional.*

H 2.2 Weyl spinors – Take I *1+1+1+2+1+2+1=9 points*

As you have probably realized the Lorentz transformation on Minkowski space is given by

$$\Lambda = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\, M^{\mu\nu}\right). \tag{8}$$

In exercise H 1.2 we have defined the Lorentz algebra through

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right). \tag{9}$$

Here we would like to investigate its representations. To make this point clear we write $D(\Lambda)$ instead of Λ.

(a) Using the notation of exercise H 1.2 we define α, β through $\omega_{ij} = \epsilon_{ijk}\alpha_k$ and $\beta_i = \omega_{0i}$. Show

$$D(\Lambda) = \exp\left(-\mathrm{i}\left[\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}\right]\right), \tag{10}$$

$$= \exp\left(-\mathrm{i}\left[\vec{\alpha} - \mathrm{i}\vec{\beta}\right]\cdot\vec{T}_{\mathrm{L}}\right)\exp\left(-\mathrm{i}\left[\vec{\alpha} + \mathrm{i}\vec{\beta}\right]\cdot\vec{T}_{\mathrm{R}}\right). \tag{11}$$

Note that T^i_{L}, T^i_{R} are still unspecified; we only know their algebra. For a particular representation one has to make a choice!

(b) Specialize to a particular representation: choose the T^i_{L}, T^i_{R} to be the Pauli matrices. The simplest representation of the Lorentz group are $(1/2, 0)$ and $(0, 1/2)$. An object transforming in the $(1/2, 0)$ is called a **left-chiral Weyl spinor**. The definition of a right-handed Weyl spinor is analogous.
How many entries does a Weyl spinor have? Write down the transformation laws for the two types of Weyl spinors.

(c) We want to rewrite the transformation laws for Weyl spinors under Lorentz transformations in the standard notation:

$$D(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\, M^{\mu\nu}\right). \tag{12}$$

Therefore, we generalize the Pauli matrices eq. (1) to

$$\sigma^\mu := \left(\mathbb{1}, \sigma^i\right), \quad \overline{\sigma}^\mu := \left(\mathbb{1}, -\sigma^i\right). \tag{13}$$

Furthermore we define the following quantities:

$$\sigma^{\mu\nu} := \frac{\mathrm{i}}{4}\left(\sigma^\mu\,\overline{\sigma}^\nu - \sigma^\nu\,\overline{\sigma}^\mu\right), \quad \overline{\sigma}^{\mu\nu} := \frac{\mathrm{i}}{4}\left(\overline{\sigma}^\mu\,\sigma^\nu - \overline{\sigma}^\nu\,\sigma^\mu\right). \tag{14}$$

We denote the left-chiral Weyl spinor $(1/2, 0)$ by Ψ_L and the right-chiral Weyl spinor $(0, 1/2)$ by Ψ_R. Let D_L, D_R denote the transformation matrices for the left- and right-chiral Weyl spinors. Show that the Weyl spinors transform as

$$\Psi_\mathrm{L} \longmapsto \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\,\sigma^{\mu\nu}\right)\Psi_\mathrm{L}\,, \tag{15}$$

$$\Psi_\mathrm{R} \longmapsto \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\,\overline{\sigma}^{\mu\nu}\right)\Psi_\mathrm{R} \tag{16}$$

Hint: Rewrite the K's and J's using the definitions T_L and T_R from exercise sheet 1. Express $M^{\mu\nu}$ in terms of K's and J's. Then identify the components of $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ with the components of $M^{\mu\nu}$.

(d) Prove the following equations:

$$D_\mathrm{L}^{-1} = D_\mathrm{R}^\dagger\,, \tag{17}$$

$$\sigma_2\,D_\mathrm{L}\,\sigma_2 = D_\mathrm{R}^*\,, \tag{18}$$

$$\sigma_2 = (D_\mathrm{L})^T\,\sigma_2\,D_\mathrm{L}\,. \tag{19}$$

Comparing the last equation to $\eta = \Lambda^T \eta\,\Lambda$, we find that σ_2 acts as a metric on the space of the spinor components!

(e) Show that $\sigma_2\Psi_\mathrm{L}^*$ transforms in the $(0, 1/2)$ representation and $\sigma_2\Psi_\mathrm{R}^*$ transforms in the $(1/2, 0)$ representation.

(f) Let Ψ_L, Ψ_R, Φ_L and Φ_R be Weyl spinors. Show that the following expressions are invariant under Lorentz transformations:

$$\mathrm{i}\,(\Phi_\mathrm{L})^T\,\sigma_2\,\Psi_\mathrm{L}\,, \tag{20}$$

$$\mathrm{i}\,(\Phi_\mathrm{R})^T\,\sigma_2\,\Psi_\mathrm{R}\,, \tag{21}$$

$$\Phi_\mathrm{R}^\dagger\Psi_\mathrm{L}\,, \tag{22}$$

$$\Phi_\mathrm{L}^\dagger\Psi_\mathrm{R}\,. \tag{23}$$

(g) Choose $\Phi_\mathrm{L} = \Psi_\mathrm{L}$ and compute $\mathrm{i}(\Psi_\mathrm{L})^T\sigma_2\,\Psi_\mathrm{L}$.
What can you conclude about spinor components?