Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 3
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Exercises on Theoretical Particle Physics

Prof. Dr. H.-P. Nilles

H 3.1 Weyl spinors – Take II

1+1+0.5+1=3.5 points

(a) Show that the parity operator acts as follows on the generators of the Lorentz algebra:

$$J^i \longmapsto J^i\,, \quad K^i \longmapsto -K^i\,. \tag{1}$$

Hint: Use $M^{\mu\nu} \mapsto \Lambda^\mu_\rho \Lambda^\nu_\sigma M^{\rho\sigma}$, *where* Λ^μ_ν *is now the parity operator.*

(b) Show that under parity transformations a representation (m, n) of the Lorentz algebra goes to (n, m), e.g. parity maps $(1/2, 0)$ to $(0, 1/2)$. Therefore, if $m \neq n$, the parity transformation maps an element of the vector space of the representation to an element that is not part of the vector space.

(c) Show that the dimension of the representation (m, n) is $(2m + 1) \cdot (2n + 1)$.

(d) Show that the 4 dim. Minkowski space is the vector space of the $(1/2, 1/2)$ representation.
Hint: Use the fact that parity maps a 4-vector to a 4-vector, i.e. you do not leave the Minkowski space if you act with parity operator.

H 3.2 Dirac spinors

1+1+1+1+1+1.5+1+0.5+2.5=10.5 points

Since the vector spaces of the left- and right-chiral Weyl spinors are not mapped to themselves under parity, we consider the following (reducible) representation of the Lorentz algebra $(1/2, 0) \oplus (0, 1/2)$. In other words: we take a left-chiral Weyl spinor Ψ_L and a right-chiral Weyl spinor Φ_R and take them as the components of a new 4-component spinor, called the Dirac spinor

$$\Psi = \begin{pmatrix} \Psi_\mathrm{L} \\ \Phi_\mathrm{R} \end{pmatrix}. \tag{2}$$

Note: We can write the Dirac spinor as two Weyl spinors in this easy way only when we use the chiral representation of the Clifford algebra.

(a) Show that the Dirac spinor transforms under a Lorentz transformation as

$$\Psi \longmapsto \Psi' = \mathfrak{D}\Psi = \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\gamma^{\mu\nu}\right)\Psi\,, \tag{3}$$

with $\gamma^{\mu\nu} := \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$. Here $\mathfrak{D}$ denotes a representation of the proper Lorentz group i.e. $\det\Lambda = +1$ and ${\Lambda^0}_0 \geq 1$. This part of the full Lorentz group contains the identity and can therefore be expressed by the exponential function.

(b) Prove the following equation

$$[\gamma^\mu, \gamma^{\nu\sigma}] = (M^{\nu\sigma})^\mu{}_\rho \, \gamma^\rho \,. \tag{4}$$

(c) Derive

$$\mathfrak{D}^{-1}\gamma^\mu\mathfrak{D} = \Lambda^\mu{}_\nu\gamma^\nu \,. \tag{5}$$

Hint: Use infinitesimal transformations $\mathfrak{D} \approx \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}$ *and* $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu$ *as well as eq. (4).*

(d) Show that in the chiral representation the chirality operator $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ can be written as

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} . \tag{6}$$

and prove that $[\gamma^5, \mathfrak{D}] = 0$

(e) Show that the following operators are a complete set of projection operators (i. e. $P^2 = P$, $P_\mathrm{L}P_\mathrm{R} = 0$, $P_\mathrm{L} + P_\mathrm{R} = \mathbb{1}$).

$$P_\mathrm{L} = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right), \quad P_\mathrm{R} = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right). \tag{7}$$

what is their action on a Dirac spinor (in the chiral representation)?

(f) Show that

$$\mathfrak{D}^\dagger = \gamma^0\mathfrak{D}^{-1}\gamma^0 \,, \tag{8}$$

and from this that follows

$$\bar{\Psi} \longmapsto \bar{\Psi}\mathfrak{D}^{-1} \,, \tag{9}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$.

(g) Consider the parity operator $\mathfrak{D}_P$, i. e. $(\Lambda_P)^0{}_0 = 1$ and $(\Lambda_P)^i{}_i = -1$. Show that one representation of the parity operator is

$$\mathfrak{D}_P = \gamma^0 \,. \tag{10}$$

Hint: Use eq. (5).

(h) Examine the action of the parity operator eq. (10) on a Dirac spinor in the chiral representation.

(i) Now we would like to analyze the list of five bilinear covariants. Check the covariance and the behavior under parity:

$$\text{scalar} \quad \bar{\Psi}\Psi \tag{11}$$
$$\text{vector} \quad \bar{\Psi}\gamma^\mu\Psi \tag{12}$$
$$\text{tensor} \quad \bar{\Psi}\gamma^{\mu\nu}\Psi \tag{13}$$
$$\text{pseudo-scalar} \quad \bar{\Psi}\gamma^5\Psi \tag{14}$$
$$\text{pseudo-vector} \quad \bar{\Psi}\gamma^5\gamma^\mu\Psi \tag{15}$$

H 3.3 Non-Abelian gauge symmetry *1+1+1+2+1+1+2+1=10 points*

(a) A Lie algebra is defined via the commutation relations of the algebra elements

$$\left[T^i, T^j\right] = \mathrm{i}\, f^{ijk} T^k \,. \tag{16}$$

The f^{ijk} are called *structure constants*. Show that the structure constants, viewed as matrices $(T^i)^{kj} := \mathrm{i} f^{ijk}$, furnish a representation of the algebra. This representation is called the *adjoint representation*.
Hint: Use the Jacobi identity.

(b) Let us take a free Dirac field Lagrangian

$$\mathscr{L} = \bar{\Psi}\left(\mathrm{i}\gamma^\mu \partial_\mu\right)\Psi \,, \tag{17}$$

with Ψ transforming under the global SU(N) as

$$\Psi \longmapsto \Psi' = U\Psi \,, \quad U = \exp\left(\mathrm{i}\,\alpha^a\, T^a\right) , \quad U^\dagger U = 1 \,. \tag{18}$$

Show that $\mathscr{L}_0$ is invariant under this transformation.

(c) As a next step wee introduce **local** SU(N) transformations.

$$\Psi \longmapsto \Psi' = U(x)\Psi \,, \quad U(x) = \exp\left(\mathrm{i}\,\alpha^a(x)\, T^a\right) , \quad U^\dagger(x)U(x) = 1 \,. \tag{19}$$

Show that the transformation of $\mathscr{L}_0$ now leads to an extra term

$$\bar{\Psi}\, U^\dagger(x)\mathrm{i}\gamma^\mu \left(\partial_\mu U(x)\right)\Psi \,. \tag{20}$$

Thus $\mathscr{L}_0$ is not invariant under local SU(N) transformations.

(d) Therefore, we want to *gauge the symmetry*: We introduce a (gauge) covariant derivative by *minimal coupling* to a gauge field and identify the gauge field's transformation properties. The covariant derivative is defined via the requirement that $D_\mu\Psi$ transforms in the same way as Ψ itself:

$$D_\mu\Psi := \left(\partial_\mu + \mathrm{i}g A^a_\mu T^a\right)\Psi \,, \tag{21}$$

demanding

$$D_\mu\Psi \longmapsto \left(D_\mu\Psi\right)' = U(x)\left(D_\mu\Psi\right) . \tag{22}$$

Show that this is equivalent to demanding that the gauge boson transforms as

$$A^a_\mu \longmapsto \left(A^a_\mu\right)' = A^a_\mu - f^{abc}\alpha^b A^c_\mu - \frac{1}{g}\partial_\mu\alpha^a \,. \tag{23}$$

Hint: Expand the exponential at the appropriate place in the calculation.

(e) Show that the following Lagrangian is gauge invariant

$$\mathscr{L} = \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu\right)\Psi \,. \tag{24}$$

(f) Define the *field strength tensor* F through

$$\mathrm{i}\, g \left(F^a_{\mu\nu} T^a\right) \Psi := \left(D_\mu D_\nu - D_\nu D_\mu\right) \Psi \tag{25}$$

and find for his components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu\,. \tag{26}$$

(g) Note that the covariant derivative was constructed such that $D'_\mu U(x) = U(x) D_\mu$ holds. Therefore

$$\left[\left(D_\mu D_\nu - D_\nu D_\mu\right) \Psi\right]' = U(x) \left(D_\mu D_\nu - D_\nu D_\mu\right) \Psi \tag{27}$$

is valid. Derive the transformation property of the field strength tensor

$$F_{\mu\nu} \longmapsto \left(F_{\mu\nu}\right)' = U\, F_{\mu\nu}\, U^{-1}\,, \tag{28}$$

$$F^a_{\mu\nu} \longmapsto \left(F^a_{\mu\nu}\right)' = F^a_{\mu\nu} - f^{abc} \alpha^b F^c_{\mu\nu}\,, \tag{29}$$

where $F_{\mu\nu} = F^a_{\mu\nu} T^a$. Because of the last equation the field strength tensor itself is not gauge invariant.

(h) Verify that the product

$$\mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) \tag{30}$$

is gauge invariant. The trace is taken over the matrix entries of the generators.

As this term is gauge invariant, we have to add it to the Lagrangian. It gives rise to self couplings of the gauge bosons. The final result for the gauge invariant Dirac Lagrangian is

$$\mathscr{L} = \bar{\Psi} \left(\mathrm{i}\gamma^\mu D_\mu\right) \Psi - \frac{1}{2} \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu}\right). \tag{31}$$