Exercises on Theoretical Particle Physics

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H 4.1 The Standard Model Higgs effect *1+2+2+1.5+2+1.5+1+1 = 12 points*

The Glashow–Weinberg–Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. In one-family approximation, the SM has the following particle content:

	$L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	$R = e_R$	$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix}$	$T^a W^a_\mu$	B_μ
Hypercharge Y	-1	-2	$+1$	0	0
$\mathrm{SU}(2)_L$ rep.	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$
Lorentz rep.	$(1/2, 0)$	$(0, 1/2)$	$(0,0)$	$(1/2, 1/2)$	$(1/2, 1/2)$

where L, R contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\begin{aligned}\mathscr{L} = &\overbrace{\overline{R}(\mathrm{i}\gamma^\mu D_\mu)R + \overline{L}(\mathrm{i}\gamma^\mu D_\mu)L}^{\text{kinetic energy terms of leptons and interactions with gauge bosons}} \quad \overbrace{-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}}^{\text{kinetic energy terms of the gauge bosons and self-interactions}} \\ &\underbrace{+(D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{\text{Higgs field with potential}} \quad \underbrace{-G_e\left(\overline{L}\Phi R + \overline{R}\Phi^\dagger L\right)}_{\text{electron–Higgs Yukawa coupling}}\,, \end{aligned} \tag{1}$$

with

$$D_\mu = \partial_\mu + \mathrm{i}g'\frac{Y}{2}B_\mu + \mathrm{i}gT^aW^a_\mu\,, \tag{2}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,. \tag{3}$$

(a) Write down how the covariant derivative eq. (2) acts on the left- and right-handed leptons doublets and on the Higgs-doublet.

(b) Show that the Lagrangian eq. (1) is Lorentz invariant.

(c) Show that eq. (1) is gauge invariant as well.

(d) For the Higgs mechanism to work we need $\mu^2 < 0$. For which value of $|\Phi|$ does the Higgs potential obtain a minimum? By an $\mathrm{SU}(2)_L$ rotation we can choose the vacuum expectation value (VEV) of the Higgs field to be of the form $\langle\Phi\rangle = \frac{1}{\sqrt{2}}\,(0, v)^T$. This leads to a redefinition of the excitation modes of the Higgs fields,

$$\Phi(x) = \exp\left\{\frac{\mathrm{i}}{v}\xi^a(x)T^a\right\}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\,(v + \eta(x)) \end{pmatrix}, \tag{4}$$

with $\xi^a(x)$ and $\eta(x)$ being real fields. Now we apply an $\mathrm{SU}(2)_L$ gauge transformation such that the angular excitations $\xi^a(x)$ vanish. This gauge transformation is called *unitary gauge*. Show that the Higgs potential in the unitary gauge is given by

$$V(\Phi) = -\mu^2\eta^2(x) + \lambda\, v\, \eta^3(x) + \frac{\lambda}{4}\eta^4(x)\,. \tag{5}$$

What is the mass of the η field? Compare the degrees of freedom (DOF) in the Higgs sector to the situation before symmetry breakdown.

(e) Consider the kinetic energy terms of the Higgs field in eq. (1). Show that

$$\begin{aligned}(D_\mu\Phi)^\dagger(D^\mu\Phi) = {}& \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{4}g^2\,(v+\eta)^2\,W^-_\mu W^{+\mu} \\ & + \frac{1}{8}\,(v+\eta)^2\begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\begin{pmatrix} g^2 & -\,g'g \\ -g'g & g'^{\,2} \end{pmatrix}\begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix},\end{aligned} \tag{6}$$

with $W^{\pm\,\mu} := \frac{1}{\sqrt{2}}(W^{1\,\mu} \mp \mathrm{i}\,W^{2\,\mu})$.

(f) The masses of the gauge bosons are given by the terms that are quadratic in the fields, e. g. $\frac{1}{4}g^2v^2W^-_\mu W^{+\mu} = m_W^2 W^-_\mu W^{+\mu}$, where $m_W = \frac{1}{2}vg$. However, to see the masses of W^3_μ and B_μ one has to diagonalize the matrix in eq. (6):

$$\frac{1}{8}\begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\mathcal{O}^T\mathcal{O}\begin{pmatrix} g^2 & -\,g'g \\ -g'g & g'^{2} \end{pmatrix}\mathcal{O}^T\mathcal{O}\begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix} = \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}. \tag{7}$$

Determine this orthogonal matrix $\mathcal{O}$ by computing the corresponding eigenvalues and eigenvectors. What are the masses of the Z_μ and A_μ fields? Compare the DOF in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of DOF?

(g) As you know, an orthogonal 2×2 matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}. \tag{8}$$

Write $\cos\theta_W$ in terms of g' and g. Show for the ratio of the W- and Z-boson masses

$$\frac{m_W}{m_Z} = \cos\theta_W\,. \tag{9}$$

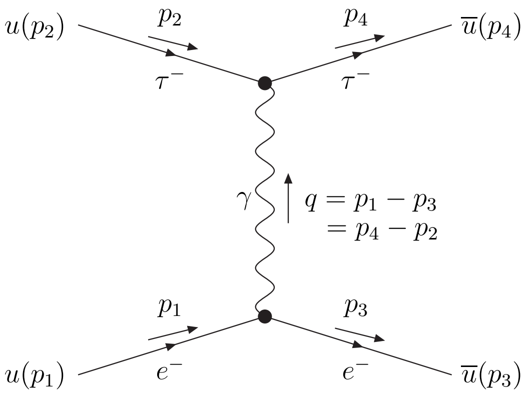
The angle θ_W is sometimes called *Weinberg angle* or *weak mixing angle*.

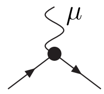
(h) Finally, consider the covariant derivative eq. (2). Substitute the fields B_μ and W^a_μ by $W^\pm_\mu$, Z_μ and A_μ and show

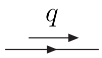
$$D_\mu = \partial_\mu + \mathrm{i}\,A_\mu\,e\,Q + \mathrm{i}Z_\mu\frac{1}{\sqrt{g'^{\,2}+g^2}}\left(g^2T_3 - g'^{\,2}\frac{Y}{2}\right) + \frac{\mathrm{i}g}{\sqrt{2}}\begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}, \tag{10}$$

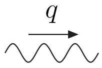
where we have defined the electric charge $e = \frac{g'g}{\sqrt{g'^{\,2}+g^2}}$ and $Q := T_3 + \frac{Y}{2}$.

H 4.2 Electron–Tauon scattering *0.5+1+1+1+1+2.5+1 = 8 points*



Vertices  $\mathrm{i}e\gamma^\mu$ (F1)

Fermions  $\mathrm{i}\frac{\not{q}+m}{q^2-m^2}$ (F2)

Photons  $-\mathrm{i}\frac{\eta_{\mu\nu}}{q^2}$ (F3)

$$\frac{\mathrm{d}\sigma(AB\to AB)}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 m_B^2}\left|\mathcal{M}\right|^2 \quad \text{(F4)}$$

In perturbative quantum field theory **Feynman Graphs** are used to calculate amplitudes of interacting processes and thus to give formulæ for cross-sections and decay widths. A Feynman graph contains **vertices** at which particles are destroyed and created, **propagators** connecting those vertices, and external lines describing in- and out-going particles.

We present the Feynman rules to calculate the amplitude $-\mathrm{i}\mathcal{M}$ in QED.

(i) An arrow in the direction of time denotes a particle, an arrow in the opposite direction denotes an antiparticle. Assign a label i to each external particle. Assign momenta to each particle (including the internal lines) and indicate them by momentum-arrows beside the particle lines.

(ii) For the following rules, proceed "backwards" with respect to the particle arrow for each fermion line. I.e. for a particle, proceeding backwards means "opposite to the direction of time". For an antiparticle, proceeding backwards means "in the direction of time".

(iii) Write a factor $u(p_i)$ ($v(p_i)$) for every external (anti-)particle line which arrow points towards a vertex and $\overline{u}(p_i)$ ($\overline{v}(p_i)$) for lines that point away from the vertex.

(iv) The contribution from vertices and internal lines (propagators) is summarized in eqs. (F1)–(F3). The indices of the γ's are contracted with the $\eta_{\mu\nu}$ of the photon proparator.

(v) Use 4-momentum conservation at the vertices to eliminate the internal momenta.

In the lab frame where the particle B is initially at rest and is assumed to be such heavy that recoil effects are negligible, the **differential cross section** for the process $AB \to AB$ is given by eq. (F4).

(a) Using the Feynman rules for QED, derive the electron-tauon scattering amplitude:

$$\mathcal{M} = -\frac{e^2}{(p_1-p_3)^2}\Big[\overline{u}(p_3)\gamma^\mu u(p_1)\Big]\Big[\overline{u}(p_4)\gamma_\mu u(p_2)\Big]. \tag{11}$$

(b) To calculate the cross section, we need to know $|\mathcal{M}|^2$. Show that

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1-p_3)^4}\Big[\overline{u}(p_3)\gamma^\mu u(p_1)\overline{u}(p_1)\gamma^\nu u(p_3)\Big]\Big[\overline{u}(p_4)\gamma_\mu u(p_2)\overline{u}(p_2)\gamma_\nu u(p_4)\Big]. \tag{12}$$

(c) In a typical experiment, the particle beam is unpolarized and the detector simply counts the number of particles scattered in a given direction. Therefore, we have to *average* over initial spins and *sum* over final spins. The averaging over the initial spins is easy: It contributes a factor of 1/2 for each sum. Using the completeness relation for Dirac spinors $\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p}+m$, where $\not{p} = p_\mu \gamma^\mu$, show that the summation over spins for the first factor in eq. (12) can be written as

$$\sum_{s_1,s_3} \overline{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\overline{u}^{(s_1)}(p_1)\gamma^\nu u^{(s_3)}(p_3) = \mathrm{tr}\Big[\,(\not{p}_3 + m_e)\,\gamma^\mu\,(\not{p}_1 + m_e)\,\gamma^\nu\Big]\,. \tag{13}$$

Derive the analogous result for the second factor in (12). The final result reads

$$\frac{1}{4}\sum_{\substack{s_1,\,s_2\\ s_3,\,s_4}} |\mathcal{M}|^2 = e^4 \frac{\mathrm{tr}\Big[(\not{p}_3 + m_e)\,\gamma^\mu\,(\not{p}_1 + m_e)\,\gamma^\nu\Big]\,\mathrm{tr}\Big[(\not{p}_4 + m_\tau)\,\gamma_\mu\,(\not{p}_2 + m_\tau)\,\gamma_\nu\Big]}{4\,(p_1 - p_3)^4}\,. \tag{14}$$

Note that we have reduced the problem of calculating the cross section to matrix multiplication and taking the trace.

(d) Consider the first trace in eq. (14). Using the identities proved in H 1.1, derive

$$\mathrm{tr}\Big[\,(\not{p}_3 + m)\,\gamma^\mu\,(\not{p}_1 + m)\,\gamma^\nu\Big] = 4\Big(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1\cdot p_3)\eta^{\mu\nu} + m_e^2\eta^{\mu\nu}\Big)\,, \tag{15}$$

and similarly for the second trace.

(e) Substitute your results in eq. (14), expand the brackets and contract the indices to show that

$$\langle|\mathcal{M}|^2\rangle = 8e^4\frac{(p_1\!\cdot\! p_2)\,(p_3\!\cdot\! p_4) + (p_1\!\cdot\! p_4)\,(p_3\!\cdot\! p_2) - (p_1\!\cdot\! p_3)\,m_\tau^2 - (p_2\!\cdot\! p_4)\,m_e^2 + 2m_\tau^2 m_e^2}{(p_1 - p_3)^4}\,. \tag{16}$$

(f) So far everything is written covariantly and is independent of the special coordinate frame. To make contact with measurements, we specify to the rest frame of the tauon and make the approximation $m_\tau \gg m_e$. Denote by $p := |\vec{p}_1|$ the absolute value of the initial electron momentum. Denote by θ the angle between $\vec{p}_1$ and $\vec{p}_3$.
Draw 2 diagrams, one before the scattering process and one after. Write the 4-momenta under the respective diagrams, taking into account the approximation we have made. Show that in this approximation conservation of energy/momentum gives $|\vec{p}_3| = |\vec{p}_1| = p$. Prove the following identities.

$$(p_1 - p_3)^2 = -4p^2\sin^2\frac{\theta}{2}\,, \qquad p_1\cdot p_3 = m_e^2 + 2p^2\sin^2\frac{\theta}{2}\,, \tag{17}$$

$$(p_1\!\cdot\! p_2)\,(p_3\!\cdot\! p_4) = E^2 m_\tau^2\,, \qquad p_2\cdot p_4 = m_\tau^2\,. \tag{18}$$

(g) Insert the above results into eq. (F4) for the cross section to obtain the **Mott formula**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{e^4}{p^4\sin^4\theta/2}\Big[m_e^2 + p^2\cos^2\theta/2\Big]\,. \tag{19}$$

In the low-energy limit this leads to the well-known **Rutherford formula**.