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Theoretische Physik

Exercise 7
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Exercises on Theoretical Particle Physics

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H 7.1 Representations of SU(N) *1+1+1+1+5.5=9.5 points*

(a) Recall the definition of the adjoint $\mathrm{ad}\, a(b) := [a, b]$.
Show that the adjoint is a representation of the Lie algebra

$$\mathrm{ad}\big([a, b]\big) = [\mathrm{ad}\, a,\ \mathrm{ad}\, b]\ , \quad \text{for } a, b \in \mathfrak{g}\,. \tag{1}$$

PLEASE NOTE!

♠ The bracket $[\cdot\,,\cdot]$ on the left-hand side denotes the abstract Lie-bracket, but on the right-hand side it denotes the commutator.

♠ The adjoint representation ad of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping ad : $\mathfrak{g} \to \mathrm{End}(V)$, where V is equal to the Lie algebra itself, i.e. $V = \mathfrak{g}$ This means that when we computed the Dynkin diagram of SU(N), we implicitly used the adjoint representation of SU(N):

$$\mathrm{ad}\, h(e_{ab}) = [h, e_{ab}]\,. \tag{2}$$

Furthermore, we had the eigenvalue equation

$$\mathrm{ad}\, h(e_{ab}) = \alpha_{e_{ab}}(h)\, e_{ab}\,, \tag{3}$$

which defined the roots $\alpha_{e_{ab}}$.
This eigenvalue equation can now be generalized to non-adjoint representations ρ on some vector space V. Let ϕ^i be a basis of V. We denote the representations of the elements of the Cartan subalgebra $h \in H$ by $\rho(h)$ and the representations of the step operators e_α by $\rho(e_\alpha)$. Then eq. (3) reads: $\rho(h)\,\phi^i = M^i(h)\,\phi^i$. Since the linear functions M^i act on elements $h \in H$ and give (real) numbers, they are elements of the dual space H^*. They are called **weights**. The corresponding vectors ϕ^i are called **weight vectors**. Note that roots are the weights of the adjoint representation!
You may have already gotten that simple roots α_j span H^*, so it is possible to reexpress the weights by simple roots $M^i = \sum_j c_{ij}\alpha_j$, where the coefficients c_{ij} are in general are in general non-integers. A weights M^i is called **positive**, if the first non-zero coefficients is positive. We write $M^i > M^j$, if $M^i - M^j > 0$.
A weight is called the **highest weight**, denoted by Λ, if $\Lambda > M^i\ \forall M^i \neq \Lambda$

(b) Suppose that ϕ^i is a weight vector with weight M^i. Show that $\rho(e_\alpha)\phi^i$ is a weight vector with weight $M^i + \alpha$ unless $\rho(e_\alpha)\phi^i = 0$.
Hint Use eqs. (2) and (3) and the fact that ρ is a representation. Thus it makes sense to think of the $\rho(e_\alpha)$ as raising operators and the $\rho(e_{-\alpha})$ as lowering operators.

(c) Consider now a representation ρ of SU(N). We denote the generators $\rho(t_a)$. For elements of the Cartan subalgebra, we may also write $\rho(h)$. Follow from

$$[\rho(t_a), \rho(t_b)] = \mathrm{i}\, f_{abc}\, \rho(t_c)\,, \tag{4}$$

that $-\rho(t_a)^*$ forms a representation, called the *complex conjugate* of ρ. We denote it by $\overline{\rho}$. ρ is said to be a real representation if it is equivalent to its complex conjugate.

(d) Show that if M^i is a weight in ρ, $-M^i$ is a weight in $\overline{\rho}$.
Hint: Use the fact that Cartan generators are hermitean and the definitions on the previous exercise sheet.

Now we are well equipped to construct the representations. For a finite dimensional representation we will find a state with highest weight Λ, which is annihilated by all positive root operators. Then we can get all states by acting with the lowering operators on it. In order to do this, we present the weights by the Dynkin labels

$$m_i := \frac{2\langle M, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\,. \tag{5}$$

where M denotes a weight. The dynkin labels always consist of integer numbers which for a highest weight state are non-negative. It is easy to see that acting with $E_{-\alpha_i}$ corresponds to substracting the ^{i}th row of the Cartan matrix from the Dynkin label. Now you can construct all irreducible representations via the following procedure:

- ✓ start with the Dynkin label m with non-negative entries, representing the highest weight state
- ✓ if the ^{i}th entry of the Dynkin label m_i is positive, you can get m_i new states by substracting m_i times the ^{i}th row of the Cartan matrix
- ✓ repeat the last step for all new steps, for $i = 1 \dots r$
- ✓ at the end you should arrive at the lowest weight state with only non-positive entries in the Dynkin label.

(e) Construct the **5** and the **10** of $\mathfrak{su}(5)$ with the highest Dynkin labels $(1,0,0,0)$ and $(0,1,0,0)$. What are the higest Dynkin labels of the $\overline{\mathbf{5}}$ and the $\overline{\mathbf{10}}$? Also, construct the adjoint, the **24**, from the Dynkin label $(1,0,0,1)$. How can you see that it is real? For an example hit the figure below.

H 7.2 Group-theoretical GUT breaking *1+1+1.5+1.5=6 points*

α_1 α_2 α_3 α_4

Dynkin Diagram of $\mathfrak{su}(5)$.

$(1,0)$

$\alpha_1 = (2,-1)$

$(-1,1)$

$\alpha_2 = (-1,2)$

$(0,-1)$

Highest weight construction. of the **3** of $\mathfrak{su}(3)$.

We believe that the SM gauge group unifies to one simple Lie algebra (e. g. $\mathfrak{su}(5)$) which is broken at very high energies $\mathcal{O}(10^{16}\,\mathrm{GeV})$. Representations os such Great Unified Theory (GUT) group decompose into those of the SM gauge group. Hence, tools for this group-theoretical symmetry breaking have to be applied.

Dynkin's Symmetry Breaking: To each simple root one assigns an integer number, called the **Kač-label** a_i. They are given as the coefficients of the decomposition of the highest root in the basis of simple roots. Deleting any node with Kac-label $a_i = 1$ from the Dynkin diagram gives a maximal regular subalgebra times a U(1) factor.

(a) In the case of SU(5), all Kač-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the SM gauge group, i. e.

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,. \tag{6}$$

The U(1) generator is constructed as a Cartan element of SU(5) such that it is annihilated by all roots of SU(3) × SU(2). Show that $Q = \mathrm{diag}(-2, -2, -2, 3, 3)$ fulfills these conditions.

(b) The **5** is a reducible representation of the subgroup SU(3) × SU(2) × U(1). Let α_1 and α_2 correspond to SU(3) and α_4 to SU(2). Thus, every weight λ of SU(5) decomposes as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\lambda_1, \lambda_2|\lambda_4) = (\mu|\nu). \tag{7}$$

First, write down all weights $(\mu|\nu)$, then find the highest weight μ and determine all weights and the dimension of the corresponding representation. Consider now the values of ν belonging to this μ-representation and state the dimension of the ν-representation! Repeat these steps starting with the highest weight ν. Finally, determine the U(1) charge by applying the U(1) generator to the weight vectors. The result reads

$$\mathbf{5} \to (\mathbf{3}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2})_3. \tag{8}$$

(c) Repeat the analysis for the representation **10** and verify

$$\mathbf{10} \to (\mathbf{1}, \mathbf{1})_6 \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{3}, \mathbf{2})_1. \tag{9}$$

Hint: All weights which appear in the calculation have multiplicity 1.

(d) Perform the breaking for the representation corresponding to the highest weight with Dynkin coefficients $(1, 0, 0, 1)$, i. e. the adjoint **24**. The result reads

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\overline{\mathbf{3}}, \mathbf{2})_5 \oplus (\mathbf{3}, \mathbf{2})_{-5}. \tag{10}$$

Identify the gauge group of the standard model.

Hint: All weights which appear in the calculation have multiplicity 1, except for $(0, 0, 0, 0)$ *in* **24** *of SU(5) with multiplicity 4 and* $(0, 0)$ *in* **8** *of SU(3) with multiplicity 2. What is the origin of this?*

After a renormalization of the U(1) generator to $Q' = \frac{1}{6}Q$ we recover one family of the standard model in $\overline{\mathbf{5}} \oplus \mathbf{10}$.

H 7.3 Dynamical GUT breaking *1.5+1+1+1=4.5 points*

It is necessary to generalize the Higgs mechanism of the SM to understand the symmetry breaking of any GUT theory to the SM. Thus, we describe the Higgs mechanism for a field H in an arbitrary representation ρ of a semi-simple Lie algebra $\mathfrak{g}$.

(a) Consider a complex scalar H in the representation ρ of a gauge group $\mathcal{G}$. Assume further that H acquires a vev $\langle H\rangle$ due to some potential. Deduce from the kinetic term[1]

$$(D_\mu H)^* (D^\mu H)\big|_{\mathbf{1}} = \left(\partial_\mu H + \mathrm{i} g\rho(T^a)A^a_\mu H\right)^* \left(\partial^\mu H + \mathrm{i} g\rho(T^b)A^{b\mu} H\right)\big|_{\mathbf{1}}\,, \qquad (11)$$

that a gauge boson A^a_μ is massless, if $\rho(T^a)\langle H\rangle = 0$. Then, T^a belongs to the unbroken gauge group $\mathcal{G}'$.

Specialize to H in the adjoint representation with the kinetic term $\mathrm{Tr}(D_\mu H)^\dagger (D^\mu H)$ and deduce

$$T^a \in \mathcal{G}' \quad \text{if} \quad [T^a, \langle H\rangle] = 0, \qquad T^a \notin \mathcal{G}' \quad \text{if} \quad [T^a, \langle H\rangle] \neq 0. \qquad (12)$$

Let us apply this for the desired symmetry breaking by introducing a Higgs field in the adjoint of SU(5), i. e. a 5×5 hermitian traceless matrix.[2] We work with a scalar potential invariant under $H \to -H$ of the form

$$V(H) = -m^2 \mathrm{Tr}\left(H^2\right) + \lambda_1 \left(\mathrm{Tr}\left(H^2\right)\right)^2 + \lambda_2 \mathrm{Tr}\left(H^4\right). \qquad (13)$$

(b) First, use the previous results to argue that a Higgs field H precisely in the adjoint **24** is an appropriate choice to break SU(5) to the SM. Which component of **24** should develope the VEV? (cf. exercise H 7.2 (d)) Use the gauge symmetry $H \to H' = UHU^\dagger$ to obtain

$$H = \mathrm{diag}(h_1, h_2, h_3, h_4, h_5) \qquad (14)$$

and check that the minimum of the potential is given by the same equation $\forall h_i$:

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2, \quad \forall i = 1, \dots, 5. \qquad (15)$$

Here μ is a Lagrange multiplier necessary to impose the constraint $\sum_i h_i = 0$.

The cubic equation (15) has at most three roots denoted by ϕ_1, ϕ_2, ϕ_3. Thus, there are at most three different eigenvalues $h_i \in \{\phi_1, \phi_2, \phi_3\}$. Let n_i be the multiplicity of the eigenvalue ϕ_i, $i = 1, 2, 3$, in $\langle H\rangle$:

$$\langle H\rangle := \mathrm{diag}(\phi_{i_1}, \dots, \phi_{i_5}) \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0. \qquad (16)$$

(c) Following part (a), what is the most general symmetry breaking of SU(5)? What happens to the rank of the gauge group? Consider also possible U(1) factors. Depending on the relative magnitude of the parameters λ_1 and λ_2, the combinations $(3, 2, 0)$ or $(4, 1, 0)$ for (n_1, n_2, n_3) minimize the potential. Thus,

$$\texttt{case1:}\ \mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,, \quad \texttt{case2:}\ \mathrm{SU}(5) \to \mathrm{SU}(4) \times \mathrm{U}(1)\,, \qquad (17)$$

which gives restrictions on phenomenologically reasonable values of λ_1, λ_2.

(d) Focus on the first case and determine what is the most general form of $\langle H\rangle$. Then, the breaking eq. (17) should be obvious. What is the generator of the U(1)? Compare this to your result for Q in exercise H 7.2 (a).

[1] The only restriction on ρ is that $\overline{\rho} \otimes \rho$ should contain **1** in order to give rise to a gauge invariant kinetic term for H. The subscript $|_{\mathbf{1}}$ denotes this singlet component. For the adjoint, it is obtained by the trace.

[2] Note that this is not the SM-Higgs field, which is contained in the **5**.