Physikalisches Institut
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Theoretische Physik

Exercise 9
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WS 09/10

Exercises on Theoretical Particle Physics

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H.9.1 Dynkin diagram of $\mathfrak{so}$(2n) *0.5+1+0.5+1+1.5+1+1.5+1.5+1.5=10 points*

The orthogonal groups are given by matrices which satisfy $A^T A = \mathbb{1}$.

(a) Using the correspondence between elements of the group and elements of the Lie algebra, $A = \exp \mathcal{A} \approx \mathbb{1} + \mathcal{A}$, show that the requirement is:

$$\mathcal{A} + \mathcal{A}^T = 0\,. \tag{1}$$

Clearly these matrices have only off-diagonal elements. As a result, it would be hard to find the Cartan subalgebra as we did for $\mathfrak{su}(n)$ by using diagonal matrices. To avoid this problem, we perform a unitary transformation on the matrices A.

(b) Use the ansatz $A = UBU^\dagger$ with U unitary, define $K = U^T U$ to show that

$$B^T K B = K\,. \tag{2}$$

Furthermore, expand B in the usual way $B = \exp \mathcal{B} \approx \mathbb{1} + \mathcal{B}$ to get the condition:

$$\mathcal{B}^T K + K\mathcal{B} = 0\,. \tag{3}$$

(c) A convenient choice for U in the case of $\mathfrak{so}(2n)$ is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i\mathbb{1} & -i\mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{pmatrix}\,, \tag{4}$$

with $\mathbb{1}$ being the $n \times n$ identity matrix. What is the form of K?

(d) We represent $\mathcal{B}$ in terms of $n \times n$ matrices $\mathcal{B}_i$:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{pmatrix}\,. \tag{5}$$

Show that from Eq.(3) follows:

$$\mathcal{B}_1 = -\mathcal{B}_4^T\,, \qquad \mathcal{B}_2 = -\mathcal{B}_2^T\,, \qquad \mathcal{B}_3 = -\mathcal{B}_3^T\,. \tag{6}$$

A basis of $2n \times 2n$ matrices fulfilling these conditions is given by $(j, k \leq n)$:

$$e^1_{jk} = e_{j,k} - e_{k+n,j+n}\,, \tag{7a}$$
$$e^2_{jk} = e_{j,k+n} - e_{k,j+n} \qquad j < k\,, \tag{7b}$$
$$e^3_{jk} = e_{j+n,k} - e_{k+n,j} \qquad j < k\,. \tag{7c}$$

A basis for the Cartan subalgebra is given by $h_j = e^1_{jj}$. So, a general element of the Cartan subalgebra can be written as:

$$h = \sum_i \lambda_i h_i\,. \tag{8}$$

(e) Determine the eigenvalues of the adjoint of h, i. e.

$$\mathrm{ad}(h)\, e^a_{jk} = [h, e^a_{jk}] = \alpha_{e^a_{jk}}(h)\, e^a_{jk}\,, \qquad a = 1, 2, 3\,. \tag{9}$$

Solution:

$$\mathrm{ad}(h)\, e^1_{jk} = (\lambda_j - \lambda_k)\, e^1_{jk} \qquad j \neq k\,, \tag{10a}$$
$$\mathrm{ad}(h)\, e^2_{jk} = (\lambda_j + \lambda_k)\, e^2_{jk} \qquad j < k\,, \tag{10b}$$
$$\mathrm{ad}(h)\, e^3_{jk} = -(\lambda_j + \lambda_k)\, e^3_{jk} \qquad j < k\,. \tag{10c}$$

Therefore, all roots are given by:

$$\alpha_{e^1_{jk}}(h) = (\lambda_j - \lambda_k) \qquad j \neq k\,, \tag{11a}$$
$$\alpha_{e^2_{jk}}(h) = (\lambda_j + \lambda_k) \qquad j < k\,, \tag{11b}$$
$$\alpha_{e^3_{jk}}(h) = -(\lambda_j + \lambda_k) \qquad j < k\,. \tag{11c}$$

(f) Convince yourself that the following roots form a basis of all roots and are furthermore positive and simple:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1}\,, \qquad i = 1 \ldots n-1\,, \tag{12}$$
$$\alpha_n(h) = \lambda_{n-1} + \lambda_n\,. \tag{13}$$

Hint: Exercise H 6.2(d)

(g) Show that the Killing form of two elements h and $h\prime$ of the Cartan subalgebra can be written in general as

$$\mathcal{K}(h, h') = 4(n-1) \sum_j \lambda_j \lambda'_j\,. \tag{14}$$

Hint: Exercise H 6.2(f)

(h) Use the theorem of exercise H 6.2 and the result of the last part to obtain from

$$\mathcal{K}(h_{\alpha_i}, h) = \alpha_i(h) \tag{15}$$

the coefficients $\lambda_j^{\alpha_i}$ of h_{α_i}.

(i) Calculate the Cartan matrix and draw the Dynkin diagram of $\mathfrak{so}(2n)$.