Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 10
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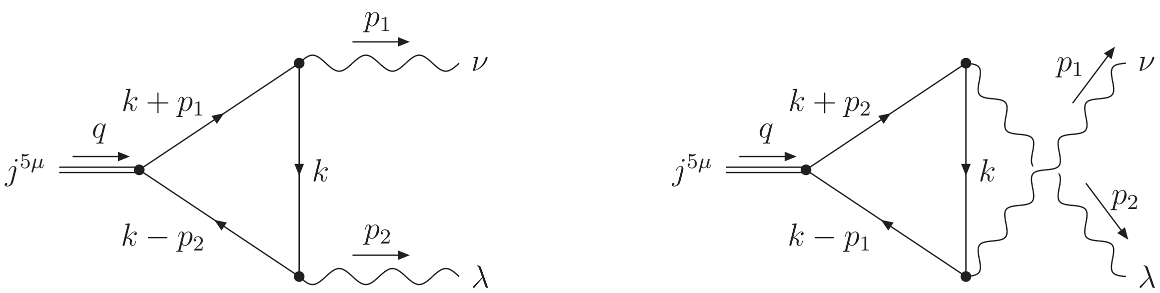
WS 09/10

Exercises on Theoretical Particle Physics

Prof. Dr. H.-P. Nilles

H 10.1 Pions are not forever

1+2+1+2+2+4+3+3=18 points



(a) First Graph

(b) Second Graph

Figure 1: Adler–Bell–Jackiw anomaly graphs.

In a classical field theory with vanishing quark masses we have a chiral symmetry implying that the axial current is conserved, $\partial_\mu j^{5\mu} = 0$. This symmetry forbids the neutral pion to decay into two photons, $\pi_0 \not\to \gamma\gamma$. However, experiment has shown that the favored decay channel of the neutral pion is the one into two photons. Therefore we must go beyond the classical level and we find that at the quantum level chiral symmetry is broken, i.e. $\partial_\mu j^{5\mu} \neq 0$. This situation in which a classical symmetry gets broken after quantization is called an **anomaly**. In this exercise we want to explore the chiral anomaly and compute the decay width of the neutral pion.

(a) Using the Feynman rules you have witnessed so far, write down the amplitude for the process in fig. 1(a). Insert for the axial vector current $\gamma^\mu\gamma^5$. Make sure you arrive at

$$\mathrm{i}\,\Pi^{\mu\nu\lambda} = -\mathrm{i}\,e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^\mu\gamma^5 \frac{\not{k} - \not{p}_2}{(k-p_2)^2}\gamma^\lambda \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k} + \not{p}_1}{(k+p_1)^2}\right]. \tag{1}$$

(b) At the classical level the matrix element of the divergence of the axial vector current vanishes. Therefore we take the divergence of eq. (1) in momentum space (i. e. we multiply with $\mathrm{i}\,q_\mu$). Furthermore, use momentum conservation to arrive at

$$\mathrm{i}\,q_\mu\,\Pi^{\mu\nu\lambda} = e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^5 \frac{\not{k} - \not{p}_2}{(k-p_2)^2}\gamma^\lambda \frac{\not{k}}{k^2}\gamma^\nu + \gamma^5\gamma^\lambda \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k} + \not{p}_1}{(k+p_1)^2}\right]. \tag{2}$$

Hint: $\not{a}\,\not{a} = a^2$

(c) As a next step we shift the integration variable in the first term of eq. (2) as $k \to k+p_2$. Using the properties of the trace, verify

$$\mathrm{i}\, q_\mu\, \Pi^{\mu\nu\lambda} = e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \mathrm{tr}\left[\gamma^5 \frac{\not{k}}{k^2}\gamma^\lambda \frac{\not{k}+\not{p}_2}{(k+p_2)^2}\gamma^\nu - \gamma^5 \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k}+\not{p}_1}{(k+p_1)^2}\gamma^\lambda\right]. \tag{3}$$

As evident, eq. (3) is antisymmetric under the interchange of (p_1, ν) and (p_2, λ). Consequently, the contribution from the second diagram in fig. 1(b) is precisely canceled and one would expect the amplitude to vanish.
However! Since we have shifted the integration variable of an divergent integral there could have appeared a finite remnant. Therefore, the shift in the integration variable we have performed requires further treatment! To do so we use a procedure called *dimensional regularization*. The basic idea is to assume that the loop momentum k has higher dimensional components while the external momenta p_1 and p_2 remain 4-dimensional. We introduce

$$k = k_4 + k_{d-4}\,, \tag{4}$$

with k_4 being the 4D Minkowski part and k_{d-4} being the higher dimensional remnant.

(d) Using the the fact that γ^5 commutes with γ^μ in the extra-dimensions (i. e. $\mu > 3$) show that the divergence of eq. (1) gets an additional contribution,

$$\mathrm{i}\, q_\mu\, \Pi^{\mu\nu\lambda} = e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \mathrm{tr}\left[-2\gamma^5\, \not{k}_{d-4} \frac{\not{k}-\not{p}_2}{(k-p_2)^2}\gamma^\lambda \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k}+\not{p}_1}{(k+p_1)^2}\right]. \tag{5}$$

Hint: Convince yourself that $q_\mu \gamma^\mu \gamma^5 = (\not{k}+\not{p}_2)\gamma^5 + \gamma^5(\not{k}-\not{p}_1) - 2\gamma^5\, \not{k}_{d-4}$.

After the dimensional regularization the shift can be justified and thus the terms in eq. (3) neatly cancel each other.

(e) Next we have to perform the Feynman trick:

$$\frac{1}{ACB} = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \frac{1}{[xA + yB + (1-x-y)C]^3}\,. \tag{6}$$

Show that the denominator becomes $[(k-\Omega)^2 - \Delta]^3$ with $\Omega = xp_2 - yp_1$ and $\Delta = x(1-x)p_2^2 + y(1-y)p_1^2 + 2xy(p_2 \cdot p_1)$.

(f) Perform the shift in the integration variable $k \to k + \Omega$. Now we look at the trace in the numerator. Note that odd powers of k vanish. Argue that the terms proportional to k^4, p_1^2 and p_2^2 vanish and the 4-dimensional parts of k drop out. Write down the remaining terms.
Hint: There are six of them.

(g) Integrate over the loop momentum. Then making use of the appropriate trace theorem and integrating over the Feynman parameters, show that the divergence of the amplitude is,

$$\mathrm{i}\, q_\mu\, \Pi^{\mu\nu\lambda} = \frac{e^2}{8\pi^2}\, \epsilon^{\nu\lambda\alpha\beta} p_{2,\alpha} p_{1,\beta}\,. \tag{7}$$

Hint: $\int \frac{d^4k}{(2\pi)^4} \frac{\not{k}_{d-4}\not{k}_{d-4}}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d-4}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)\Delta^{2-d/2}}$.
Taking the limit $d \to 4$ *this integral becomes* $\frac{-i}{2(4\pi)^2}$.

Obviously eq. (7) is symmetric under the interchange of (p_1, ν) and (p_2, λ) and therefore the second diagram in fig. 1(a) gives an equal contribution.
To obtain the amplitude for the decay $\pi^0 \to 2\gamma$ from both diagrams we can use our result eq. (7). We make the ansatz

$$\mathrm{i}\,\Pi = \mathrm{i}\,\frac{e^2}{4\pi^2}\frac{1}{f_\pi}\varepsilon_\nu^{(i)*}\varepsilon_\lambda^{(j)*}\epsilon^{\nu\lambda\alpha\beta}p_{2,\alpha}p_{1,\beta}\,, \tag{8}$$

where $\varepsilon^{(i)*}$ are the polarization vectors and f_π denotes the *pion decay constant* which parameterizes the QCD effects.

(h) Finally, using the Hans–Josef Formula we can compute the width of the pion decay:

$$\Gamma(\pi^0 \to 2\gamma) = \frac{1}{2m_\pi}\frac{1}{8\pi}\frac{1}{2}\sum_{\substack{\text{polarization}\\ \text{states}}}\left|\Pi(\pi^0 \to 2\gamma)\right|^2\,, \tag{9}$$

where m_π denotes the mass of the neutral pion and the factor $1/2$ is due to phase space of identical particles. Useful data: $m_\pi = 134.976\,\mathrm{MeV}$ and $f_\pi = 93\,\mathrm{MeV}$. Compare your result to the experimentally measured value $\Gamma^{(\mathrm{exp})} = 7.93\,\mathrm{eV}$ and/or the mean life-time $\tau^{(exp)} = 8.4 \times 10^{-17}\,\mathrm{s}$

Hint: Go to the rest frame of the pion and use momentum conservation and Lorentz invariance to determine the form of $p_{2,\alpha}$ *and* $p_{1,\beta}$. *Remember that the photon has two polarization states which are orthogonal to its momentum. Recall:* $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$.