Physikalisches Institut
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Theoretische Physik

Exercise 0
20. October 2010
WS 10/11

Exercises on Theoretical Particle Physics

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–Class Exercises–

C 1.1 The Little Group

In this exercise we want to explore the little group under which particle states transform. A general group action of a group G on a set X will be denoted by

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \,. \end{aligned}$$

Then the little group of an element $x \in X$ is the set of transformations which leaves x invariant, i.e.

$$G_x := \{g \in G \mid gx = x\} \,.$$

(a) Show that G_x is indeed a subgroup of G.

(b) In particle physics we are interested in the little group of the momentum of a particle as a subgroup of the Lorentz group. Denoting by $p^\mu \in \mathbb{R}^{1,3}$ the momentum of a four dimensional particle, we find the condition

$$\Lambda^\mu{}_\nu p^\nu = p^\mu \,, \qquad \Lambda \in SO(1,3) \,.$$

How does this condition translate to the Lie algebra $\mathfrak{so}(1,3)$?

(c) A basis of $\mathfrak{so}(1,3)$ is given by the matrices

$$(M^{\mu\nu})^\rho{}_\sigma = \mathrm{i}\left(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma\right) \,.$$

Now we look at a massive particle. Its momentum can be rotated to the form $p^\mu = (m, 0, 0, 0)$. Which generators leave p^μ invariant? What is the little group of a massive state?

(d) The momentum of a massless particle can be chosen $p^\mu = (p, 0, 0, p)$. Find three (linear combinations of) generators which leave p^μ invariant. Describe the corresponding group action. The group they generate is isomorphic to the so-called Euclidean group $E(2)$.

(e) Show that two of these three generators correspond to non-compact directions by explicitly computing the group elements. Find the maximal compact subgroup of $E(2)$. Since non-trivial irreducible representations of non-compact groups are infinite dimensional, we restrict the little group of massless particles to the maximal compact subgroup by projecting the states onto their representations.

C 1.2 Massive and massless Vector Bosons.
In this exercise we want to apply the things we just learned about little groups to the particular example of a vector boson. Consider first a massive vector boson A_μ described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{2}A^\mu A_\mu \,,$$

where as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Derive the equations of motion

$$\partial^\mu \partial_\mu A^\nu - \partial^\nu \partial_\mu A^\mu + m^2 A^\nu = 0 \,.$$

(b) Deduce from the e.o.m. that every component satisfies the Klein–Gordon equation and that there is an additional condition of the form $\partial_\mu A^\mu = 0$. This condition reduces the number of degrees of freedom from four to three as required for a vector representation of $SO(3)$.

(c) Now we consider the massless case $m = 0$. Show that the massless Lagrangian is invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi$.

(d) Show that we can use this gauge freedom to fulfill the Lorentz gauge condition $\partial_\mu A^\mu = 0$. How do the equations of motion then look like? *Hint: Use the Greens function of the d'Alembert operator* $\Box$.

(e) For a massless particle state $A_\mu(x) = \epsilon_\mu e^{\mathrm{i}q_\nu x^\nu}$ there is more freedom in the choice of a gauge. Show that a gauge transformation of the form $\chi(x) = c e^{\mathrm{i}q_\nu x^\nu}$ does not spoil the Lorentz gauge condition. How does this transformation act on the polarization vector ϵ_μ?

(f) Choose $q_\nu = (q, 0, 0, q)$. Use c to set $\epsilon_0 = 0$. How does the Lorentz gauge condition further restrict ϵ_μ? Find a basis of the remaining two-dimensional space of physical photon polarizations.