Physikalisches Institut
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Theoretische Physik

Exercise 2
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WS 10/11

Exercises on Theoretical Particle Physics

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–Home Exercises–
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H 2.1 Weyl spinors

1+1+1+2+1+2+1+1+1+0.5+1=12.5 points

As you have probably realized the Lorentz transformation on Minkowski space is given by

$$\Lambda = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\, M^{\mu\nu}\right) .$$

In exercise H 1.1 we have defined the Lorentz algebra through

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right) .$$

Here we would like to investigate its representations. To make this point clear we write $D(\Lambda)$ instead of Λ.

(a) Using the notation of exercise H 1.1 we define $\vec{\alpha}$, $\vec{\beta}$ through $\omega_{ij} = \epsilon_{ijk}\alpha_k$ and $\beta_i = \omega_{0i}$. Show

$$\begin{aligned} D(\Lambda) &= \exp\left(-\mathrm{i}\left[\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}\right]\right) , \\ &= \exp\left(-\mathrm{i}\left[\vec{\alpha} - \mathrm{i}\vec{\beta}\right]\cdot\vec{T}_{\mathrm{L}}\right)\exp\left(-\mathrm{i}\left[\vec{\alpha} + \mathrm{i}\vec{\beta}\right]\cdot\vec{T}_{\mathrm{R}}\right) . \end{aligned}$$

Note that T^i_{L}, T^i_{R} are still unspecified; we only know their algebra. For a particular representation one has to make a choice!

(b) Specialize to a particular representation: choose the T^i_{L}, T^i_{R} to be the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

The simplest representations of the Lorentz group are $(1/2, 0)$ and $(0, 1/2)$. An object transforming in the $(1/2, 0)$ is called a **left-chiral Weyl spinor**. The definition of a right-handed Weyl spinor is analogous.
How many entries does a Weyl spinor have? Write down the transformation laws for the two types of Weyl spinors.

(c) We want to rewrite the transformation laws for Weyl spinors under Lorentz transformations in the standard notation:

$$D(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\, M^{\mu\nu}\right).$$

Therefore, we generalize the Pauli matrices eq. (1) to

$$\sigma^\mu := \left(\mathbb{1}, \sigma^i\right), \quad \overline{\sigma}^\mu := \left(\mathbb{1}, -\sigma^i\right).$$

Furthermore we define the following quantities:

$$\sigma^{\mu\nu} := \frac{\mathrm{i}}{4}\left(\sigma^\mu\, \overline{\sigma}^\nu - \sigma^\nu\, \overline{\sigma}^\mu\right), \quad \overline{\sigma}^{\mu\nu} := \frac{\mathrm{i}}{4}\left(\overline{\sigma}^\mu\, \sigma^\nu - \overline{\sigma}^\nu\, \sigma^\mu\right).$$

We denote the left-chiral Weyl spinor $(1/2, 0)$ by Ψ_{L} and the right-chiral Weyl spinor $(0, 1/2)$ by Ψ_{R}. Let D_{L}, D_{R} denote the transformation matrices for the left- and right-chiral Weyl spinors. Show that the Weyl spinors transform as

$$\Psi_{\mathrm{L}} \longmapsto \exp\left(-\frac{\mathrm{i}}{2}\, \omega_{\mu\nu}\, \sigma^{\mu\nu}\right) \Psi_{\mathrm{L}}\,,$$
$$\Psi_{\mathrm{R}} \longmapsto \exp\left(-\frac{\mathrm{i}}{2}\, \omega_{\mu\nu}\, \overline{\sigma}^{\mu\nu}\right) \Psi_{\mathrm{R}}$$

Hint: Rewrite the K's and J's using the definitions T_{L} and T_{R} from exercise sheet 1. Express $M^{\mu\nu}$ in terms of K's and J's. Then identify the components of $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ with the components of $M^{\mu\nu}$.

(d) Prove the following equations:

$$D_{\mathrm{L}}^{-1} = D_{\mathrm{R}}^{\dagger}\,,$$
$$\sigma_2\, D_{\mathrm{L}}\, \sigma_2 = D_{\mathrm{R}}^{*}\,,$$
$$\sigma_2 = \left(D_{\mathrm{L}}\right)^T \sigma_2\, D_{\mathrm{L}}\,.$$

Comparing the last equation to $\eta = \Lambda^T \eta\, \Lambda$, we find that σ_2 acts as a metric on the space of the spinor components!

(e) Show that $\sigma_2 \Psi_{\mathrm{L}}^*$ transforms in the $(0, 1/2)$ representation and $\sigma_2 \Psi_{\mathrm{R}}^*$ transforms in the $(1/2, 0)$ representation.

(f) Let Ψ_{L}, Ψ_{R}, Φ_{L} and Φ_{R} be Weyl spinors. Show that the following expressions are invariant under Lorentz transformations:

$$\mathrm{i}\left(\Phi_{\mathrm{L}}\right)^T \sigma_2\, \Psi_{\mathrm{L}}\,,$$
$$\mathrm{i}\left(\Phi_{\mathrm{R}}\right)^T \sigma_2\, \Psi_{\mathrm{R}}\,,$$
$$\Phi_{\mathrm{R}}^\dagger \Psi_{\mathrm{L}}\,,$$
$$\Phi_{\mathrm{L}}^\dagger \Psi_{\mathrm{R}}\,.$$

(g) Choose $\Phi_{\mathrm{L}} = \Psi_{\mathrm{L}}$ and compute $\mathrm{i}(\Psi_{\mathrm{L}})^T \sigma_2\, \Psi_{\mathrm{L}}$. What can you conclude about spinor components?

(h) Show that the parity operator acts as follows on the generators of the Lorentz algebra:

$$J^i \longmapsto J^i\,, \quad K^i \longmapsto -K^i\,.$$

Hint: Use $M^{\mu\nu} \mapsto \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$*, where* Λ *is now the parity operator.*

(i) Show that under parity transformations a representation (m, n) of the Lorentz algebra goes to (n, m), e.g. parity maps $(1/2, 0)$ to $(0, 1/2)$. Therefore, if $m \neq n$, the parity transformation maps an element of the vector space of the representation to an element that is not part of the vector space.

(j) Show that the dimension of the representation (m, n) is $(2m + 1) \cdot (2n + 1)$.

(k) Show that the 4 dim. Minkowski space is the vector space of the $(1/2, 1/2)$ representation.
Hint: Use the fact that parity maps a 4-vector to a 4-vector, i.e. you do not leave the Minkowski space if you act with parity operator.

H 2.2 Dirac spinors *1+1+1+1+1+1.5+1+0.5+2.5=10.5 points*

Since the vector spaces of the left- and right-chiral Weyl spinors are not mapped to themselves under parity, we consider the following (reducible) representation of the Lorentz algebra $(1/2, 0) \oplus (0, 1/2)$. In other words: we take a left-chiral Weyl spinor Ψ_L and a right-chiral Weyl spinor Φ_R and take them as the components of a new 4-component spinor, called the Dirac spinor

$$\Psi = \begin{pmatrix} \Psi_\mathrm{L} \\ \Phi_\mathrm{R} \end{pmatrix}.$$

Note: We can write the Dirac spinor as two Weyl spinors in this easy way only when we use the chiral representation of the Clifford algebra.

(a) Show that the Dirac spinor transforms under a Lorentz transformation as

$$\Psi \longmapsto \Psi' = \mathfrak{D}\Psi = \exp\left(-\frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\gamma^{\mu\nu}\right)\Psi\,,$$

with $\gamma^{\mu\nu} := \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$ and γ^μ in the Weyl representation.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Here $\mathfrak{D}$ denotes a representation of the proper Lorentz group i.e. $\det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$. This part of the full Lorentz group contains the identity and can therefore be expressed by the exponential function.

(b) Prove the following equation

$$[\gamma^\mu, \gamma^{\nu\sigma}] = (M^{\nu\sigma})^\mu{}_\rho\, \gamma^\rho\,. \tag{2}$$

(c) Derive

$$\mathfrak{D}^{-1}\gamma^\mu\mathfrak{D} = \Lambda^\mu{}_\nu\gamma^\nu\,. \tag{3}$$

Hint: Use infinitesimal transformations $\mathfrak{D} \approx \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}$ *and* $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu$ *as well as eq. (2).*

(d) Show that in the chiral representation the chirality operator $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ can be written as

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

and prove that $[\gamma^5, \mathfrak{D}] = 0$

(e) Show that the following operators are a complete set of projection operators (i. e. $P^2 = P$, $P_\mathrm{L}P_\mathrm{R} = 0$, $P_\mathrm{L} + P_\mathrm{R} = \mathbb{1}$).

$$P_\mathrm{L} = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right), \quad P_\mathrm{R} = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right).$$

what is their action on a Dirac spinor (in the chiral representation)?

(f) Show that

$$\mathfrak{D}^\dagger = \gamma^0\mathfrak{D}^{-1}\gamma^0\,,$$

and from this that follows

$$\bar{\Psi} \longmapsto \bar{\Psi}\mathfrak{D}^{-1}\,,$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$.

(g) Consider the parity operator $\mathfrak{D}_P$, i. e. $(\Lambda_P)^0{}_0 = 1$ and $(\Lambda_P)^i{}_i = -1$. Show that one representation of the parity operator is

$$\mathfrak{D}_P = \gamma^0\,. \tag{4}$$

Hint: Use eq. (3).

(h) Examine the action of the parity operator eq. (4) on a Dirac spinor in the chiral representation.

(i) Now we would like to analyze the list of five bilinear covariants. Check the covariance and the behavior under parity:

$$\begin{array}{rl} \text{scalar} & \bar{\Psi}\Psi \\ \text{vector} & \bar{\Psi}\gamma^\mu\Psi \\ \text{tensor} & \bar{\Psi}\gamma^{\mu\nu}\Psi \\ \text{pseudo-scalar} & \bar{\Psi}\gamma^5\Psi \\ \text{pseudo-vector} & \bar{\Psi}\gamma^5\gamma^\mu\Psi \end{array}$$