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Exercise 3
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Exercises on Theoretical Particle Physics

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–Home Exercises–
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H 3.1 Representations of $\mathfrak{su}(2)$ *1+1.5+1+1+1+1.5+0.5+1.5+2=11 points*

A Lie algebra $\mathfrak{g}$ is a real vector space together with a smooth map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

(i) The map is bilinear.

(ii) The map is skew-symmetric: $[a, b] = -[b, a]$ for $a, b \in \mathfrak{g}$.

(iii) It fulfills the Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for $a, b, c \in \mathfrak{g}$.

A *representation* ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ which is an algebra homomorphism, i. e. $\rho([a, b]) = [\rho(a), \rho(b)]$. The dimension of V is called the dimension of the representation: $\dim(\rho) := \dim(V)$.
If there is a vector space $\{0\} \neq W \subsetneq V$ such that $\rho(W) \subset W$, the representation is called reducible and W is called the invariant subspace. If such W does not exist the representation is called *irreducible*; i. e. a representation is irreducible *if and only if* V is the only invariant subspace itself. In this exercise we will focus on the algebra $\mathfrak{su}(2)$.

(a) For $G \in \mathrm{SU}(2)$ we can write $G = e^{\mathrm{i}\, g}$ with $g \in \mathfrak{su}(2)$. The group SU(2) is the set of all 2-dimensional unitary matrices with determinant 1. Show that the corresponding Lie algebra $\mathfrak{su}(2)$ is the set of all traceless hermitian matrices.
Hint: $\det A = \exp \mathrm{Tr} \log A$.

(b) Choose the basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for the traceless hermitian matrices with the commutation relation $[\sigma^i, \sigma^j] = 2\mathrm{i}\epsilon^{ijk}\sigma^k$ and define

$$J_3 = \frac{1}{2}\sigma_3, \quad J_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2), \quad J_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2).$$

Verify the commutation relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3.$$

In the following let us consider **all irreducible, finite-dimensional representations** of $\mathfrak{su}(2)$ on a vector space V, $\rho(J_i) \in \mathrm{End}(V)$, $i = 3, +, -$. We will proceed stepwise in order to classify these representations and to find out which $\dim(V)$ are allowed.

(c) Since J_3 is diagonal, $\rho(J_3)$ can also be chosen to be diagonal. Therefore V can be decomposed into eigenspaces of $\rho(J_3)$,

$$V = \bigoplus V_\alpha \,,$$

where α labels the eigenvalues of $\rho(J_3)$, i. e.

$$(\rho(J_3))\, v = \alpha v, \quad v \in V_\alpha \,, \quad \alpha \in \mathbb{C} \,.$$

Show that $J_+(v) \in V_{\alpha+1}$ and $J_-(v) \in V_{\alpha-1}$.

Hint: For convenience use the shorthand notation J_i for $\rho(J_i)$.

(d) Prove that all complex eigenvalues α which appear in the above decomposition differ from one another by 1.

Hint: Choose an arbitrary $\alpha_0 \in \mathbb{C}$ from the decomposition and prove that $\bigoplus_{k\in\mathbb{Z}} V_{\alpha_0+k} \subset V$ is indeed equal to V using the irreducibility of the representation.

(e) Argue that there is a $k \in \mathbb{N}$ for which $V_{\alpha_0+k} \neq \{0\}$ and $V_{\alpha_0+k+1} = \{0\}$. Define $n := \alpha_0 + k$. Note that up to now we only know that $n \in \mathbb{C}$. Draw a diagram by writing the vector spaces V_{n-2}, V_{n-1} and V_n in a row and indicating the action J_3, J_+ and J_- on these vector spaces by arrows. The eigenvalue n is called *highest weight* and a vector $v \in V_n$ is called *highest weight vector*. Why?

(f) Choose an arbitrary vector $v \in V_n$ (highest weight vector). Prove that the vectors v, J_-v, $J_-^2 v$, ... span V.

Hint: Show that the vector space spanned by these vectors is invariant under the action of J_3, J_+ and J_- and use the irreducibility of the representation.

(g) Argue that all eigenspaces V_α are 1-dimensional.

(h) Prove that n is a non-negative integer or half integer and that

$$V = V_{-n} \otimes \ldots \otimes V_n \,.$$

Complement the diagram drawn in part (e). Which is the dimension of the representation?

Hint: The representation is finite dimensional, so there exists $m \in \mathbb{N}$ for which $J_-^{m-1} v \neq 0$ and $J_-^m v = 0$. Evaluate $J_+ J_-^m v$.

(i) Consider the tensor product of a 2-dimensional and a 3-dimensional irreducible representations of $\mathfrak{su}(2)$:

$$V = V^{(2)} \otimes V^{(3)} \,.$$

Show that the resulting representation V is reducible and that it can be decomposed into a 2-dim. and a 4-dim. irreducible representation. Shorthand: $\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{4}$.

Hint: The action of a Lie algebra on the tensor product of two representations is given by: $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, i. e. the eigenvalue of J_3 on V is the sum of *the eigenvalues of J_3 on $V^{(2)}$ and $V^{(3)}$. Draw the diagrams of the eigenvalues(with multiplicities). Then use the fact that the eigenspaces of irreducible representations are 1-dimensional.*

H 3.2 Non-Abelian gauge symmetry *1+1+1+2+1+1+2+1=10 points*

(a) A Lie algebra is defined via the commutation relations of the algebra elements

$$\left[T^i, T^j\right] = \mathrm{i}\, f^{ijk} T^k \,.$$

The f^{ijk} are called *structure constants*. Show that the structure constants, viewed as matrices $(T^i)^{kj} := \mathrm{i} f^{ijk}$, furnish a representation of the algebra. This representation is called the *adjoint representation*.
Hint: Use the Jacobi identity.

(b) Let us take a free Dirac field Lagrangian

$$\mathscr{L}_0 = \bar{\Psi}\left(\mathrm{i}\gamma^\mu \partial_\mu\right)\Psi \,,$$

with Ψ transforming under the global $\mathrm{SU}(N)$ as

$$\Psi \longmapsto \Psi' = U\Psi \,, \quad U = \exp\left(\mathrm{i}\,\alpha^a\, T^a\right), \quad U^\dagger U = 1 \,.$$

Show that $\mathscr{L}_0$ is invariant under this transformation.

(c) As a next step we introduce **local** $\mathrm{SU}(N)$ transformations.

$$\Psi \longmapsto \Psi' = U(x)\Psi \,, \quad U(x) = \exp\left(\mathrm{i}\,\alpha^a(x)\, T^a\right), \quad U^\dagger(x) U(x) = 1 \,.$$

Show that the transformation of $\mathscr{L}_0$ now leads to an extra term

$$\bar{\Psi}\, U^\dagger(x) \mathrm{i}\gamma^\mu \left(\partial_\mu U(x)\right)\Psi \,.$$

Thus $\mathscr{L}_0$ is not invariant under local $\mathrm{SU}(N)$ transformations.

(d) Therefore, we want to *gauge the symmetry*: We introduce a (gauge) covariant derivative by *minimal coupling* to a gauge field and identify the gauge field's transformation properties. The covariant derivative is defined via the requirement that $D_\mu \Psi$ transforms in the same way as Ψ itself:

$$D_\mu \Psi := \left(\partial_\mu + \mathrm{i} g A_\mu^a T^a\right)\Psi \,,$$

demanding

$$D_\mu \Psi \longmapsto \left(D_\mu \Psi\right)' = U(x)\left(D_\mu \Psi\right).$$

Show that this is equivalent to demanding that the gauge boson transforms as

$$A_\mu^a \longmapsto \left(A_\mu^a\right)' = A_\mu^a - f^{abc} \alpha^b A_\mu^c - \frac{1}{g}\partial_\mu \alpha^a \,.$$

Hint: Expand the exponential at the appropriate place in the calculation.

(e) Show that the following Lagrangian is gauge invariant

$$\mathscr{L} = \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu\right)\Psi\,.$$

(f) Define the *field strength tensor* F through

$$\mathrm{i}\,g\left(F^a_{\mu\nu}T^a\right)\Psi := \left(D_\mu D_\nu - D_\nu D_\mu\right)\Psi$$

and find for its components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,f^{abc}A^b_\mu A^c_\nu\,.$$

(g) Note that the covariant derivative was constructed such that $D'_\mu U(x) = U(x)D_\mu$ holds. Therefore

$$\left[\left(D_\mu D_\nu - D_\nu D_\mu\right)\Psi\right]' = U(x)\left(D_\mu D_\nu - D_\nu D_\mu\right)\Psi$$

is valid. Derive the transformation property of the field strength tensor

$$\begin{aligned} F_{\mu\nu} &\longmapsto \left(F_{\mu\nu}\right)' = U\,F_{\mu\nu}\,U^{-1}\,, \\ F^a_{\mu\nu} &\longmapsto \left(F^a_{\mu\nu}\right)' = F^a_{\mu\nu} + f^{abc}\alpha^b F^c_{\mu\nu}\,, \end{aligned}$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a$. Because of the last equation the field strength tensor itself is not gauge invariant.

(h) Verify that the product

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

is gauge invariant. The trace is taken over the matrix entries of the generators.

As this term is gauge invariant, we have to add it to the Lagrangian. It gives rise to self couplings of the gauge bosons. The final result for the gauge invariant Dirac Lagrangian is

$$\mathscr{L} = \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu\right)\Psi - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right).$$