Physikalisches Institut
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Theoretische Physik

Exercise 4
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WS 10/11

Exercises on Theoretical Particle Physics

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H 4.1 Gell-Mann Matrices *0.5+6+3+2.5=12 points*

The standard basis for the fundamental representation of $\mathfrak{su}(3)$ is

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Why are there exactly 8 matrices in the basis?

(b) Evaluate the commutators of these matrices to determine the structure constants f^{abc}. Show that, with the normalizations used here, f^{abc} is totally antisymmetric. *Hint: This exercise is tedious; you may which to check only a representative sample of the commutators.*

(c) Check the orthogonality relation,

$$\mathrm{tr}\, T^a T^b = C(r)\delta^{ab},$$

where $C(r)$ is a constant for each representation r. Evaluate the constant $C(r)$ for this representation. *Hint: Just check it for a representative sample. What is the trace of a product of a diagonal and a off-diagonal matrix?*

(d) Why is $\{T^3, T^8\}$ a good choice for the Cartan subalgebra? Show that it is diagonalized by the (complex) basis transformation,

$$T_\pm = T^1 \pm \mathrm{i}T^2, \qquad U_\pm = T^4 \pm \mathrm{i}T^5, \qquad V_\pm = T^6 \pm \mathrm{i}T^7.$$

H 4.2 The Standard Model Higgs effect *1+2+2+1.5+2+1.5+1+1 = 12 points*

The Glashow–Weinberg–Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. In a one-family approximation, the SM has the following particle content:

	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$R = e_R$	$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$T^a W^a_\mu$	B_μ
Hypercharge Y	-1	-2	$+1$	0	0
$\mathrm{SU}(2)_L$ rep.	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$
Lorentz rep.	$(1/2, 0)$	$(0, 1/2)$	$(0,0)$	$(1/2, 1/2)$	$(1/2, 1/2)$

where L, R contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\begin{aligned}\mathscr{L} = &\overbrace{\overline{R}(\mathrm{i}\gamma^\mu D_\mu)R + \overline{L}(\mathrm{i}\gamma^\mu D_\mu)L}^{\text{kinetic energy terms of leptons and interactions with gauge bosons}} \quad \overbrace{-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}}^{\text{kinetic energy terms of the gauge bosons and self-interactions}} \\ &\underbrace{+(D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{\text{Higgs field with potential}} \quad \underbrace{-G_e\left(\overline{L}\Phi R + \overline{R}\Phi^\dagger L\right)}_{\text{electron–Higgs Yukawa coupling}}\,, \end{aligned} \tag{1}$$

with

$$D_\mu = \partial_\mu + \mathrm{i}g'\frac{Y}{2}B_\mu + \mathrm{i}gT^aW^a_\mu\,, \tag{2}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,. \tag{3}$$

(a) Write down how the covariant derivative eq. (2) acts on the left- and right-handed leptons doublets and on the Higgs-doublet.

(b) Show that the Lagrangian eq. (1) is Lorentz invariant.

(c) Show that eq. (1) is gauge invariant as well.

(d) For the Higgs mechanism to work we need $\mu^2 < 0$. For which value of $|\Phi|$ does the Higgs potential obtain a minimum? By an $\mathrm{SU}(2)_L$ rotation we can choose the vacuum expectation value (VEV) of the Higgs field to be of the form $\langle\Phi\rangle = \frac{1}{\sqrt{2}}(0, v)^T$. This leads to a redefinition of the excitation modes of the Higgs fields,

$$\Phi(x) = \exp\left\{\frac{\mathrm{i}}{v}\xi^a(x)T^a\right\}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}\,,$$

with $\xi^a(x)$ and $\eta(x)$ being real fields and T^a the generators of SU(2). Now we apply an $\mathrm{SU}(2)_L$ gauge transformation such that the angular excitations $\xi^a(x)$ vanish. This gauge transformation is called *unitary gauge*. Show that the Higgs potential in the unitary gauge is given by

$$V(\Phi) = -\mu^2\eta^2(x) + \lambda\, v\, \eta^3(x) + \frac{\lambda}{4}\eta^4(x)\,.$$

What is the mass of the η field? Compare the degrees of freedom (DOF) in the Higgs sector to the situation before symmetry breakdown.

(e) Consider the kinetic energy terms of the Higgs field in eq. (1). Show that

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) = \frac{1}{2}\partial_\mu \eta\, \partial^\mu \eta + \frac{1}{4} g^2 (v+\eta)^2\, W_\mu^- W^{+\mu} + \frac{1}{8}(v+\eta)^2 \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix}, \qquad (4)$$

with $W^{\pm\mu} := \frac{1}{\sqrt{2}}(W^{1\mu} \mp \mathrm{i}\, W^{2\mu})$.

(f) The masses of the gauge bosons are given by the terms that are quadratic in the fields, e.g. $\frac{1}{4}g^2 v^2 W_\mu^- W^{+\mu} = m_W^2 W_\mu^- W^{+\mu}$, where $m_W = \frac{1}{2} v g$. However, to see the masses of W_μ^3 and B_μ one has to diagonalize the matrix in eq. (4):

$$\frac{1}{8}\begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \mathcal{O}^T \mathcal{O} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \mathcal{O}^T \mathcal{O} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} = \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}.$$

Determine this orthogonal matrix $\mathcal{O}$ by computing the corresponding eigenvalues and eigenvectors. What are the masses of the Z_μ and A_μ fields? Compare the DOF in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of DOF?

(g) As you know, an orthogonal 2×2 matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}.$$

Write $\cos\theta_W$ in terms of g' and g. Show for the ratio of the W- and Z-boson masses

$$\frac{m_W}{m_Z} = \cos\theta_W\,.$$

The angle θ_W is sometimes called *Weinberg angle* or *weak mixing angle.*

(h) Finally, consider the covariant derivative eq. (2). Substitute the fields B_μ and W_μ^a by $W_\mu^\pm$, Z_μ and A_μ and show

$$D_\mu = \partial_\mu + \mathrm{i}\, A_\mu\, e\, Q + \mathrm{i} Z_\mu \frac{1}{\sqrt{g'^2 + g^2}} \left(g^2 T_3 - g'^2 \frac{Y}{2} \right) + \frac{\mathrm{i} g}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix},$$

where we have defined the electric charge $e = \frac{g'g}{\sqrt{g'^2+g^2}}$ and $Q := T_3 + \frac{Y}{2}$.