Exercises on Theoretical Particle Physics

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–Home Exercises–
Due 26. November 2010

H 6.1 More on the CKM matrix *6 points*

In this exercise we want to focus a bit more on the precise structure of the CKM matrix and investigate its physical content. Let $M_{ij} = \frac{v}{\sqrt{2}} G_{ij}$ be the mass matrix in an N flavour model after spontaneous symmetry breaking. We assume it to have full rank.

(a) Show that $MM^\dagger$ is Hermitean. Thus it can be diagonalized by a unitary matrix S, i.e.

$$S^\dagger MM^\dagger S = M_d^2 = \mathrm{diag}(m_1^2, \ldots, m_N^2)\,,$$

where for a physically stable vacuum we choose $m_i > 0$. We can rewrite this as

$$MM^\dagger = SM_d^2 S^\dagger\,.$$

show that the right hand side of this equation has N more free parameters than a Hermitean matrix (like $MM^\dagger$). Show that this leaves us the freedom to transform $S \to SF$ with $F = \mathrm{diag}\left(e^{\mathrm{i}\phi_1}, \ldots, e^{\mathrm{i}\phi_N}\right)$. *(1 point)*

(b) Define a Hermitean matrix by $H = SM^d S^\dagger$. Show that $V := H^{-1}M$ is unitary. *(1 point)*

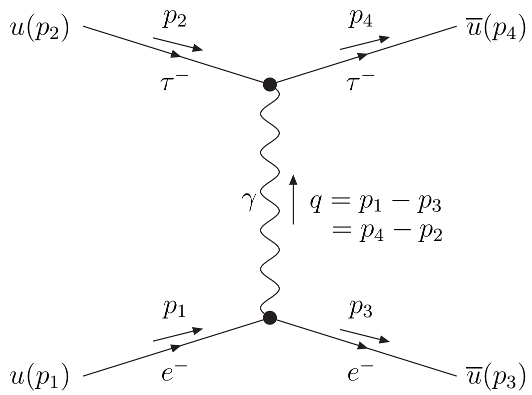
(c) Show that this allows us to write $M = SM_d T^\dagger$ with $T = V^\dagger S$ also unitary. Compare again the number of parameters of the general matrix M to the number of parameters in $SM_d T^\dagger$. Identify the same freedom in choice of the matrices S, T given by the matrix F. *(1 point)*

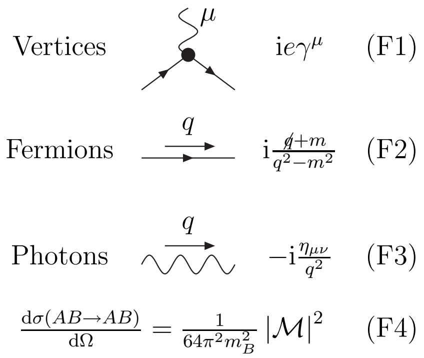
(d) Remember that the CKM matrix is defined by $V_{\mathrm{CKM}} = U_u^\dagger U_d$ where the biunitary transformations acting on the quark mass matrices are $M_i \to V_i^\dagger U_i$ for $i = u, d$, see last sheet. Thus it is a unitary matrix. Show that using the freedom to choose F_u and F_d, V_{CKM} has $(N-1)^2$ physical parameters. *Hint: Identify a one-parameter subgroup within the F_u and F_d which does not change V_{CKM}.* *(1.5 points)*

(e) In the framework of $U(N)$ these parameters can be interpreted as mixing angles which are the same as in $SO(N)$ and complex phases. Show that the amount of complex phases for N generations is $(N-1)(N-2)/2$. *(1 point)*

(f) Physical complex phases in the CKM matrix lead to CP violating processes. What is the minimal amount of families required to observe CP violation as was done in K^0 decays? *(0.5 points)*

H 6.2 Electron–Muon scattering *11 points*





In perturbative quantum field theory **Feynman Graphs** are used to calculate amplitudes of interacting processes and thus to give formulæ for cross-sections and decay widths. A Feynman graph contains **vertices** at which particles are destroyed and created, **propagators** connecting those vertices, and external lines describing in- and out-going particles.

We present the Feynman rules to calculate the amplitude $-\mathrm{i}\mathcal{M}$ in QED.

(i) An arrow in the direction of time denotes a particle, an arrow in the opposite direction denotes an antiparticle. Assign a label i to each external particle. Assign momenta to each particle (including the internal lines) and indicate them by momentum-arrows beside the particle lines.

(ii) For the following rules, proceed "backwards" with respect to the particle arrow for each fermion line. I.e. for a particle, proceeding backwards means "opposite to the direction of time". For an antiparticle, proceeding backwards means "in the direction of time".

(iii) Write a factor $u(p_i)$ $(v(p_i))$ for every external (anti-)particle line which arrow points towards a vertex and $\overline{u}(p_i)$ $(\overline{v}(p_i))$ for lines that point away from the vertex.

(iv) The contribution from vertices and internal lines (propagators) is summarized in eqs. (F1)–(F3). The indices of the γ's are contracted with the $\eta_{\mu\nu}$ of the photon proparator.

(v) Use 4-momentum conservation at the vertices to eliminate the internal momenta.

In the lab frame where the particle B is initially at rest and is assumed to be such heavy that recoil effects are negligible, the **differential cross section** for the process $AB \to AB$ is given by eq. (F4).

(a) Using the Feynman rules for QED, derive the electron-muon scattering amplitude:

$$\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2}\Big[\overline{u}(p_3)\gamma^\mu u(p_1)\Big]\Big[\overline{u}(p_4)\gamma_\mu u(p_2)\Big].$$

(0.5 points)

(b) To calculate the cross section, we need to know $|\mathcal{M}|^2$. Show that

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4}\Big[\overline{u}(p_3)\gamma^\mu u(p_1)\overline{u}(p_1)\gamma^\nu u(p_3)\Big]\Big[\overline{u}(p_4)\gamma_\mu u(p_2)\overline{u}(p_2)\gamma_\nu u(p_4)\Big]. \qquad (1)$$

(1.5 points)

(c) In a typical experiment, the particle beam is unpolarized and the detector simply counts the number of particles scattered in a given direction. Therefore, we have to *average* over initial spins and *sum* over final spins. The averaging over the initial spins is easy: It contributes a factor of 1/2 for each sum. Using the completeness relation for Dirac spinors $\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p}+m$, where $\not{p} = p_\mu \gamma^\mu$, show that the summation over spins for the first factor in eq. (1) can be written as

$$\sum_{s_1,s_3} \overline{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\overline{u}^{(s_1)}(p_1)\gamma^\nu u^{(s_3)}(p_3) = \mathrm{tr}\Big[\,(\not{p}_3 + m_e)\,\gamma^\mu\,(\not{p}_1 + m_e)\,\gamma^\nu\Big]\,.$$

Derive the analogous result for the second factor in (1). The final result reads

$$\frac{1}{4}\sum_{\substack{s_1,\,s_2\\ s_3,\,s_4}} |\mathcal{M}|^2 = e^4 \frac{\mathrm{tr}\Big[(\not{p}_3 + m_e)\,\gamma^\mu\,(\not{p}_1 + m_e)\,\gamma^\nu\Big]\,\mathrm{tr}\Big[(\not{p}_4 + m_\mu)\,\gamma_\mu\,(\not{p}_2 + m_\mu)\,\gamma_\nu\Big]}{4\,(p_1 - p_3)^4}\,. \qquad (2)$$

Note that we have reduced the problem of calculating the cross section to matrix multiplication and taking the trace. *(1.5 points)*

(d) Consider the first trace in eq. (2). Using the identities proved in H 1.1, derive

$$\mathrm{tr}\Big[\,(\not{p}_3 + m)\,\gamma^\mu\,(\not{p}_1 + m)\,\gamma^\nu\Big] = 4\Big(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1\cdot p_3)\eta^{\mu\nu} + m_e^2\eta^{\mu\nu}\Big)\,,$$

and similarly for the second trace. *(1.5 points)*

(e) Substitute your results in eq. (2), expand the brackets and contract the indices to show that

$$\langle|\mathcal{M}|^2\rangle = 8e^4 \frac{(p_1\!\cdot\! p_2)\,(p_3\!\cdot\! p_4) + (p_1\!\cdot\! p_4)\,(p_3\!\cdot\! p_2) - (p_1\!\cdot\! p_3)\,m_\mu^2 - (p_2\!\cdot\! p_4)\,m_e^2 + 2m_\mu^2 m_e^2}{(p_1 - p_3)^4}\,.$$

(1.5 points)

(f) So far everything is written covariantly and is independent of the special coordinate frame. To make contact with measurements, we specify to the rest frame of the muon and make the approximation $m_\mu \gg m_e$. Denote by $p := |\vec{p}_1|$ the absolute value of the initial electron momentum. Denote by θ the angle between $\vec{p}_1$ and $\vec{p}_3$.
Draw 2 diagrams, one before the scattering process and one after. Write the 4-momenta under the respective diagrams, taking into account the approximation we have made. Show that in this approximation conservation of energy/momentum gives $|\vec{p}_3| = |\vec{p}_1| = p$. Prove the following identities.

$$(p_1 - p_3)^2 = -4p^2 \sin^2\frac{\theta}{2}\,, \qquad p_1\cdot p_3 = m_e^2 + 2p^2\sin^2\frac{\theta}{2}\,,$$
$$(p_1\!\cdot\! p_2)\,(p_3\!\cdot\! p_4) = E^2 m_\mu^2\,, \qquad p_2\cdot p_4 = m_\mu^2\,.$$

(3 points)

(g) Insert the above results into eq. (F4) for the cross section to obtain the **Mott formula**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{e^4}{p^4\sin^4\theta/2}\Big[m_e^2 + p^2\cos^2\theta/2\Big]\,.$$

In the low-energy limit this leads to the well-known **Rutherford formula**. *(1.5 points)*