Physikalisches Institut
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Theoretische Physik

Exercise 7
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WS 10/11

Exercises on Theoretical Particle Physics

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–Home Exercises–

Due 3. December 2010

H 7.1 Majorana spinors and the See–Saw mechanism *10 points*

We write a four component Dirac spinor in the chiral representation as a composition of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix} .$$

A *Majorana spinor* is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\overline{\Psi}^T = \Psi \,, \tag{1}$$

where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation operator.

(a) Show that $(\Psi^c)^c = \Psi$. (*0.5 points*)

(b) What does eq. (1) imply for ψ_L and ψ_R and what is the physical meaning of this condition? (*1 point*)

(c) The Lagrangian $\mathscr{L}_\mathrm{D}$ for a Dirac spinor has the form

$$\mathscr{L}_\mathrm{D} = \overline{\Psi}\left(\mathrm{i}\gamma^\mu\partial_\mu\right)\Psi - m\overline{\Psi}\Psi \,,$$

where the second term is called the *Dirac mass term*. Rewrite $\mathscr{L}_\mathrm{D}$ in ψ_L and ψ_R. (*1 point*)

(d) Using the result of (b) rewrite the Lagrangian $\mathscr{L}_\mathrm{M}$ for a Majorana spinor in terms of $\psi_{\mathrm{L/R}}$

$$\mathscr{L}_\mathrm{M} = \overline{\Psi}\left(\mathrm{i}\gamma^\mu\partial_\mu\right)\Psi - \frac{m}{2}\overline{\Psi}\Psi.$$

The second term is called the *Majorana mass term*. Why is the factor 1/2 included in the mass term? (*1 point*)

(e) Remember the projectors $P_{\mathrm{L/R}} = 1/2(\mathbb{1} \mp \gamma^5)$. As you know $P_{\mathrm{L/R}}$ project Ψ onto the left/right handed part, respectively. We denote $\Psi_{\mathrm{L/R}} := P_{\mathrm{L/R}}\Psi$. Show that

$$\left(\Psi_{\mathrm{L/R}}\right)^c = (\Psi^c)_{\mathrm{R/L}}$$
$$\overline{\left(\Psi_{\mathrm{L/R}}\right)^c}\left(\Psi_{\mathrm{R/L}}\right)^c = \overline{\Psi_{\mathrm{R/L}}}\Psi_{\mathrm{L/R}}$$

(*1 point*)

(f) The most general mass term for a Dirac spinor is the *Dirac–Majorana mass term*,

$$\mathscr{L}_{\mathrm{m}} = -\frac{1}{2}\left[2m_{\mathrm{D}}\overline{\Psi}\Psi + m_{\mathrm{L}}\overline{\Psi_{\mathrm{L}}}(\Psi^c)_{\mathrm{R}} + m_{\mathrm{R}}\overline{(\Psi^c)_{\mathrm{L}}}\Psi_{\mathrm{R}}\right]$$

Show that this can be written in matrix form as

$$\mathscr{L}_{\mathrm{m}} = -\frac{1}{2}\begin{pmatrix}\overline{\Psi_{\mathrm{L}}} & \overline{(\Psi^c)_{\mathrm{L}}}\end{pmatrix}\mathcal{M}\begin{pmatrix}(\Psi^c)_{\mathrm{R}}\\ \Psi_{\mathrm{R}}\end{pmatrix}$$

with

$$\mathcal{M} = \begin{pmatrix} m_{\mathrm{L}} & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_{\mathrm{R}}\end{pmatrix}$$

being the *neutrino mass matrix*. (*1 point*)

(g) Argue that in the SM extended by right-handed neutrinos, m_{L} must be zero and m_{D} is of the order of the electroweak symmetry breaking scale $M_W \sim 100\,\mathrm{GeV}$. We further assume that m_{R} is generated by some unspecified symmetry breaking mechanisms occurring at high energies, i. e. $m_{\mathrm{R}} \sim M_{\mathrm{GUT}} \sim 10^{16}\,\mathrm{GeV}$. (*1 point*)

(h) In this setup, diagonalize $\mathcal{M}$ using an orthogonal matrix A

$$A^T\mathcal{M}A = \mathrm{diag}(m_1, m_2).$$

Show that to the first non-vanishing order in the (small) parameter $\rho := m_{\mathrm{D}}/m_{\mathrm{R}}$ that the eigenvalues are $m_1 = -m_{\mathrm{D}}^2/m_{\mathrm{R}}$ and $m_2 = m_{\mathrm{R}}$. Find the rotation matrix A to the first order in ρ for the diagonalization. What does $\rho \ll 1$ imply for the mass eigenstates? Insert the estimations done in (g) and compare the mass of the light neutrino to actual experimental bounds. (*3.5 points*)

We see that by making one mass heavy the other one becomes very light. For this reason setups of this kind are generically referred to as *See-Saw mechanism*.

H 7.2 Neutrino oscillations *6 points*

In analogy to the mixing of the quarks through weak interactions via the CKM matrix one can imagine a similar situation with the leptons once the neutrinos get mass. Hence, let us assume that there are n orthonormal flavor (interaction) eigenstates $|\nu_\alpha\rangle$. These states are transformed into n mass eigenstates ν_i via the unitary mixing matrix U,

$$|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle \,.$$

(a) Assuming that the mass eigenstates $|\nu_{\mathrm{i}}\rangle$ are stationary states and were emitted with momentum p by a source at $x = 0$ at $t = 0$, what is the form of $|\nu_i(x,t)\rangle$? (*0.5 points*)

(b) What is the relativistic Hamiltonian for a particle? For a highly relativistic particle we have $m \ll p$. Expand the Hamiltonian to first non-vanishing order in m/p. (*1 point*)

(c) A neutrino detector is built at a distance L from a source producing neutrinos in an eigenstate $|\nu_\alpha\rangle$. Show that the amplitude of detecting a neutrino in an eigenstate $|\nu_\beta\rangle$ is

$$A(\alpha \to \beta)(L) = \sum_i U^*_{\beta i} U_{\alpha i} \exp\left\{ \mathrm{i}\, \frac{m_i^2}{2} \frac{L}{E} \right\} \,.$$

Hint: For a highly relativistic particle you can set $v = 1 (= c)$ and $p \cong E$. (*1.5 points*)

(d) Obtain the transition probability P in terms of the differences of the mass squares $\Delta m_{ij}^2 := m_i^2 - m_j^2$. What is the probability of finding the original flavor? (*1 point*)

(e) Now assume that we have two flavors and one mixing angle θ. What is the form of U? Compute $P(\alpha \to \beta)$ and $P(\alpha \to \alpha)$ for this case. Under which condition can one flavor completely rotate into another one? (*2 points*)