# Exercises on Theoretical Particle Physics

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## –Home Exercises–
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### H 8.1 Dynkin diagram of $\mathfrak{su}(N)$ *11 points*

Consider the space of all $N \times N$ matrices and regard it as a Lie algebra $\mathfrak{gl}(N)$. We choose as a basis the elements $e_{ab}$ with components $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

(a) Verify the multiplication rule and thus the commutator operation on the algebra

$$e_{ab}e_{cd} = e_{ad}\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad}.$$

In order to deal with the Lie algebra $\mathfrak{su}(N)$, what restrictions have to be made? Write down a basis for $\mathfrak{su}(N)$. What is the dimension? (*1,5 points*)

(b) The **Cartan algebra** $\mathfrak{h}$ is defined to be a maximal commuting subalgebra of the Lie algebra. Its dimension is called the **rank** of the Lie algebra. Give a possible choice for the Cartan subalgebra of $\mathfrak{su}(N)$. What is the rank $r$ of $\mathfrak{su}(N)$? (*1 point*)

(c) Now we want to diagonalize the Cartan algebra in the adjoint representation which acts by the commutator

$$\mathrm{ad}\, h(g) = [h, g]$$

Perform a (complex) basis change of $\mathfrak{su}(N)/\mathfrak{h}$ to an eigenbasis of $\mathfrak{h}$. You should find,

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab}\,, \tag{1}$$

with $h = \sum_i \lambda_i e_{ii}$. (*0.5 points*)

We can regard eq. (1) (for $e_{ab}$ fixed) as a prescription for how to associate a number $(\lambda_a - \lambda_b)$ to each $h \in \mathfrak{h}$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b.$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra $\mathfrak{h}$. This dual space is denoted by $\mathfrak{h}^*$.
Let $\alpha_1 \dots \alpha_r$ be a fixed basis of roots so every element of $\mathfrak{h}^*$ can be written as $\rho = \sum_i c_i\alpha_i$. We call $\rho$ **positive** ($\rho > 0$) if the first non-zero coefficient $c_i$ is positive. Note, that the basis roots $\alpha_i$ are positive by definition. If the first non-zero coefficient $c_i$ is negative, we call $\rho$ negative. For $\rho, \sigma \in \mathfrak{h}^*$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$. A **simple root** is a positive root which can not be written as the sum of two positive roots.

(d) We choose a basis $\alpha_i$ for the root space:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1}, \quad i = 1, 2, \ldots, N-1\,.$$

Verify that these roots are a basis and that they are positive with $\alpha_1 > \alpha_2 > \ldots > \alpha_{N-1}$. Show that these roots are simple roots. *(1.5 points)*

Next, we define a structure that resembles a scalar product on the algebra. Let $t_i$ be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra:

$$[x, [y, t_i]] = \sum_j K_{ij} t_j\,.$$

The **Killing form** is then defined as $\mathcal{K}(x, y) := \mathrm{Tr}(K)$.

(e) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h, h')$, where $h = \sum_i \lambda_i e_{ii}$, $h' = \sum_j \lambda'_j e_{jj}$. *(1.5 points)*

The Killing form enables us to make a connection between the Cartan subalgebra, $\mathfrak{h}$, and its dual $\mathfrak{h}^*$: One can prove that if $\alpha \in \mathfrak{h}^*$, there exists a unique element $h_\alpha \in \mathfrak{h}$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \quad \forall h \in \mathfrak{h}\,.$$

(f) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find $h_{\alpha_i}$ from comparison with your result from (e). *(1 point)*

With the help of the $h_\alpha$, we are now able to define a scalar product on $\mathfrak{h}^*$:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \quad \text{where } \alpha_i, \alpha_j \in \mathfrak{h}^*.$$

(g) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}\,.$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root $\alpha_i$, we associate a small circle and join the small circles $i$ and $j$ with $A_{ij}A_{ij}$ (no summation, $i \neq j$) lines. *(1.5 points)*

(h) Draw the Dynkin diagram for $\mathfrak{su}(N)$. *(0.5 points)*

(i) As an example, consider the Lie algebra of $\mathfrak{su}(2)$. The step operators are given by

$$J_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)\,,$$

and the Cartan subalgebra consists of the single element

$$h = J_3 = \frac{1}{2}\sigma_3\,.$$

(i.1) Confirm that

$$e_{12} = J_+, \quad e_{21} = J_- \quad \text{and} \quad h = \frac{1}{2}e_{11} - \frac{1}{2}e_{22} \,.$$

(i.2) Calculate $\alpha_{J_\pm}(J_3)$.

(i.3) Choose $\alpha_1 = \alpha_{J_+}$ as the basis root, which is positive and simple. For $\alpha_1 \in \mathfrak{h}^*$, find the unique element $h_{\alpha_1} \in \mathfrak{h}$ such that

$$\alpha_1(h) = \mathcal{K}(h_{\alpha_1}, h) \quad \forall h \in \mathfrak{h} \,.$$

*Hint: The solution is* $h_{\alpha_1} = \frac{1}{2}J_3$.

(i.4) Calculate the Killing form $\mathcal{K}(h_{\alpha_1}, h_{\alpha_2})$ and draw the Dynkin diagram. (*2 points*)

**H 8.2 Representations of $\mathfrak{su}(N)$** *9 points*

(a) Recall the definition of the adjoint $\operatorname{ad} a(b) := [a, b]$.
Show that the adjoint is a representation of the Lie algebra

$$\operatorname{ad}\big([a, b]\big) = [\operatorname{ad} a, \, \operatorname{ad} b] \;, \quad \text{for } a, b \in \mathfrak{g} \,.$$

(*1 point*)

**PLEASE NOTE!**

♣ The bracket $[\cdot\,, \cdot]$ on the left-hand side denotes the abstract Lie-bracket, but on the right-hand side it denotes the commutator.

♣ The adjoint representation ad of a Lie algebra $\mathfrak{g}$ on a vector space $V$ is a linear mapping ad : $\mathfrak{g} \to \operatorname{End}(V)$, where $V$ is equal to the Lie algebra itself, i.e. $V = \mathfrak{g}$ This means that when we computed the Dynkin diagram of SU($N$), we implicitly used the adjoint representation of SU($N$):

$$\operatorname{ad} h(e_{ab}) = [h, e_{ab}] \,. \tag{2}$$

Furthermore, we had the eigenvalue equation

$$\operatorname{ad} h(e_{ab}) = \alpha_{e_{ab}}(h)\, e_{ab} \,, \tag{3}$$

which defined the roots $\alpha_{e_{ab}}$.
This eigenvalue equation can now be generalized to non-adjoint representations $\rho$ on some vector space $V$. Let $\phi^i$ be a basis of $V$. We denote the representations of the elements of the Cartan subalgebra $h \in H$ by $\rho(h)$ and the representations of the step operators $e_\alpha$ by $\rho(e_\alpha)$. Then eq. (3) reads: $\rho(h)\,\phi^i = M^i(h)\,\phi^i$. Since the linear functions $M^i$ act on elements $h \in H$ and give (real) numbers, they are elements of the dual space $H^*$. They are called **weights**. The corresponding vectors $\phi^i$ are called **weight vectors**. Note that roots are the weights of the adjoint representation!
You may have already gotten that simple roots $\alpha_j$ span $H^*$, so it is possible to reexpress the weights by simple roots $M^i = \sum_j c_{ij}\alpha_j$, where the coefficients $c_{ij}$ are in general are in general non-integers. A weights $M^i$ is called **positive**, if the first non-zero coefficients is positive. We write $M^i > M^j$, if $M^i - M^j > 0$.
A weight is called the **highest weight**, denoted by $\Lambda$, if $\Lambda > M^i \;\forall M^i \neq \Lambda$

(b) Suppose that $\phi^i$ is a weight vector with weight $M^i$. Show that $\rho(e_\alpha)\phi^i$ is a weight vector with weight $M^i + \alpha$ unless $\rho(e_\alpha)\phi^i = 0$.
*Hint Use eqs. (2) and (3) and the fact that $\rho$ is a representation. Thus it makes sense to think of the $\rho(e_\alpha)$ as raising operators and the $\rho(e_{-\alpha})$ as lowering operators.* (*1 point*)

(c) Consider now a representation $\rho$ of SU($N$). We denote the generators $\rho(t_a)$. For elements of the Cartan subalgebra, we may also write $\rho(h)$. Follow from

$$[\rho(t_a), \rho(t_b)] = \mathrm{i}\, f_{abc}\, \rho(t_c)\,,$$

that $-\rho(t_a)^*$ forms a representation, called the *complex conjugate* of $\rho$. We denote it by $\overline{\rho}$. $\rho$ is said to be a real representation if it is equivalent to its complex conjugate. (*1 point*)

(d) Show that if $M^i$ is a weight in $\rho$, $-M^i$ is a weight in $\overline{\rho}$.
*Hint: Use the fact that Cartan generators are hermitean and the definitions on the previous exercise sheet.* (*1 point*)

Now we are well equipped to construct the representations. For a finite dimensional representation we will find a state with highest weight $\Lambda$, which is annihilated by all positive root operators. Then we can get all states by acting with the lowering operators on it. In order to do this, we present the weights by the Dynkin labels

$$m_i := \frac{2\langle M, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\,.$$

where $M$ denotes a weight. The dynkin labels always consist of integer numbers which for a highest weight state are non-negative. It is easy to see that acting with $E_{-\alpha_i}$ corresponds to substracting the $^i$th row of the Cartan matrix from the Dynkin label. Now you can construct all irreducible representations via the following procedure:

- ◇ start with the Dynkin label $m$ with non-negative entries, representing the highest weight state
- ◇ if the $^i$th entry of the Dynkin label $m_i$ is positive, you can get $m_i$ new states by substracting $m_i$ times the $^i$th row of the Cartan matrix
- ◇ repeat the last step for all new steps, for $i = 1 \ldots r$
- ◇ at the end you should arrive at the lowest weight state with only non-positive entries in the Dynkin label.

(e) Construct the **5** and the **10** of $\mathfrak{su}(5)$ with the highest Dynkin labels $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. What are the higest Dynkin labels of the $\overline{\mathbf{5}}$ and the $\overline{\mathbf{10}}$? Also, construct the adjoint, the **24**, from the Dynkin label $(1, 0, 0, 1)$. How can you see that it is real? (*5 points*)