Exercises on String Theory I

Prof. Dr. H.P. Nilles

–Home Exercises–
Due 15. November 2011

Exercise 3.1: Convergence of the Zeta Function *10 Credits*

To calculate the normal ordering constant a in the vacuum energy in the canonical quantization of a string theory, one finds series corresponding to certain values of the Hurwitz zeta funcion $\zeta(s,b)$. In this exercise we perform analytic manipulations to compute the values of $\zeta(s,b)$ at the points of interest. First of all we define

$$\zeta(s,b) = \sum_{n=0}^{\infty} \frac{1}{(n+b)^s}\,. \tag{1}$$

1. Show the following identities:

$$\frac{1}{\nu^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\nu t} t^{s-1} \mathrm{d}t\,, \quad \text{with} \quad \Gamma(s) := \int_0^\infty e^{-t} t^{s-1} \mathrm{d}t\,, \tag{2}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (b+k)^{1-s} = \frac{1}{\Gamma(s-1)} \int_0^\infty e^{-bt} \left(1-e^{-t}\right)^n t^{s-2} \mathrm{d}t\,, \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(1-e^{-t})^n}{n+1} = \frac{t}{1-e^{-t}}\,. \tag{4}$$

(*3 credits*)

2. Use the formulae above to show that

$$\zeta(s,b) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (b+k)^{1-s}\,. \tag{5}$$

(*4 credits*)

3. Show that

$$\zeta(-1,b) = -\frac{b(b-1)}{2} - \frac{1}{12}\,. \tag{6}$$

In particular we get $\zeta(-1,0) = -\frac{1}{12}$.
Hint: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^l$ *vanishes for* $0 \le l < n$ (*3 credits*)

Exercise 3.2: Gamma Matrices *10 Credits*

In D dimensions, the Clifford algebra is given by D matrices Γ^μ which satisfy

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\,.$$

Here, $\mu = 0, \ldots, D-1$ and $\eta = \mathrm{diag}(-, +, \ldots, +)$.

1. Show that the matrices

   $$\Sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\,[\Gamma^\mu, \Gamma^\nu]$$

   form a representation of the Lorentz algebra.

   $$[J^{\mu\nu}, J^{\rho\sigma}] = (\mathrm{i}\eta^{\mu\rho}J^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma)\,.$$

   This representation is called spinor representation, and the elements of the representation space are (Dirac) spinors. (*2 credits*)

2. Define a new matrix Γ_* by

   $$\Gamma_* = \mathrm{i}^\alpha \Gamma^0 \cdots \Gamma^{D-1}\,.$$

   α is a parameter to be determined later. Show that Γ_* (anti)commutes with the Γ^μ,

   $$\{\Gamma_*, \Gamma^\mu\} = 0 \quad \text{for } D \text{ even}\ , \qquad [\Gamma_*, \Gamma^\mu] = 0 \quad \text{for } D \text{ odd.}$$

   (Note that this implies that for odd D, Γ_* is a multiple of the unit matrix.)

   Show that $\Gamma_*^2 \sim \mathbb{1}$, and find (for even D) an α such that $\Gamma_*^2 = \mathbb{1}$. (*2 credits*)

3. For even D, define the operators $P_\pm = \frac{1}{2}\,(1 \pm \Gamma_*)$. Verify that they form a complete set of orthogonal projectors. These projectors define right- and left-chiral (Weyl) spinors.

   Prove that the representation of the Lorentz group by the generators $\Sigma^{\mu\nu}$ is reducible. To do so, show that it splits into two mutually commuting representations generated by the chiral generators $\Sigma_+^{\mu\nu} = \Sigma^{\mu\nu}P_+$ and $\Sigma_-^{\mu\nu}$. (*2 credits*)

4. Consider a spinor $\psi = (\psi_1, \psi_2)^T$ in $D = 2$. The Γ matrices are given by

   $$\gamma^0 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

   Find the Lorentz generator Σ^{01}. Determine the action of the Lorentz group by $\exp\{\mathrm{i}\omega_{01}\Sigma^{01}\}$ on ψ, where ω_{01} is a real parameter. How does the Lorentz group act on the chiral components of the spinor? (*2 credits*)

5. A Majorana condition is a reality condition on the spinor of the form

   $$\psi^* = B\psi$$

   with some invertible matrix B. Show that consistency requires $BB^* = 1$ and $B\Sigma^{01}B^{-1} = -\Sigma^{01*}$.

   Find a matrix B that works and show that it is compatible with chirality, i.e. that the reality condition can be imposed on the chiral components. (*2 credits*)