Exercises on String Theory I

Prof. Dr. H.P. Nilles

–Home Exercises–
Due 06. December 2011

In String Theory, it is conjectured that there exists an 11 dimensional theory, called M–Theory, from which the different 10 dimensional String Theories arise. Although the full description of M–Theory is unknown, the low energy effective theory has been identified as 11 dimensional supergravity (SUGRA). Upon compactification on a circle, one can get to the 10 dimensional Type IIA supergravity description.
On this exercise sheet, we work out this reduction. In the first exercise, we examine the dimensional reduction of a fermion. In the second exercise, we investigate the reduction when the metric (i.e. gravity) is included. In the last exercise, we then examine the dimensional reduction of 11 dimensional SUGRA to 10 dimensional Type IIA SUGRA.

Exercise 6.1: Kaluza Klein reduction of Fermions (*6 credits*)

We want to calculate the spectrum of a Fermion which we dimensionally reduce on a circle. For this, we first investigate higher dimensional gamma matrices. They satisfy the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\,,$$

where we use the convention $\eta^{\mu\nu} = \mathrm{diag}(-,+,\ldots,+)$, $\mu,\nu = 0,\ldots,D-1$. Furthermore, we define $\Gamma^* = (-1)^{-\frac{D(D-1)}{2}-1}\,\Gamma^0\cdot\ldots\cdot\Gamma^{D-1}$ and $P_{L/R} = \frac{1}{2}(\mathbb{1}\pm\Gamma^*)\,.$

(a) In $D+1$ dimensions Γ^* plays the role of Γ^D. We start with a $D+1$ dimensional Dirac theory given by

$$\mathcal{L} = \int \mathrm{d}x^{D+1}\bar{\Psi}\Gamma^N\partial_N\Psi + M_{D+1}\bar{\Psi}\Psi\,,$$

where $N = 0,\ldots,D$, $\bar{\Psi} = \Psi^\dagger\Gamma^0$. We compactify the theory on a circle of radius R, i.e. $x^D \sim x^D + 2\pi R$. Make an ansatz for Ψ in terms of eigenfunctions of ∂_D with periodic boundary conditions and with Ψ split into $\Psi_{L/R} = P_{L/R}\Psi$. What is the mass matrix for the momentum number modes? Diagonalize it. The resulting Kaluza Klein tower is the same as for the Kaluza Klein scalar. (*5 credits*)

(b) Show that for $M_{D+1} = 0$ there are two chiral spinors of opposite chirality in D dimensions. (*1 credit*)

Exercise 6.2: Kaluza–Klein reduction including Gravity (*7 credits*)

For a compactification of gravity from $D+1$ to D dimensions we make the ansatz

$$G_{MN} = \phi^\beta \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}, \qquad G^{MN} = \phi^{-\beta} \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \phi^{-1} + A_\rho A^\rho \end{pmatrix}.$$

The $D+1$ dimensional Einstein Hilbert action is

$$S_{D+1} = \frac{1}{16\pi\kappa_{D+1}} \int \mathrm{d}^{D+1}x \, \sqrt{-G}\, R_{D+1}$$

(a) Determine the exponent β as a function of D, by requiring that one obtains the D dimensional Einstein Hilbert action after compactification on a circle

$$S_{D+1} \to S_D = \frac{1}{16\pi\kappa_D} \int \mathrm{d}^D x \, \sqrt{-g}\, R_D + \ldots$$

The choice of β for which Φ does not appear explicitly is called Einstein Frame. *Hint: Do NOT perform the full dimensional reduction. Just compute* $\sqrt{-G}$ *and assume that* $\phi = \mathrm{const}$ *in* R_{D+1}. *Then* β *follows from cancelation of* ϕ *in front of* R_D. (*4 credits*)

(b) What is the D dimensional Newton constant κ_D in terms of the radius R? (*1 credit*)

(c) Show that $D+1$ diemnsional general coordinate transformations

$$G_{MN} \to \frac{\partial x^R}{\partial x'^M} \frac{\partial x^S}{\partial x'^N} G_{RS},$$

induce D dimensional gauge transformations when reparametrizing the circle coordinate as $x^D \to x^D + \lambda(x^\mu)$, $x^\mu \to x^\mu$. (*2 credits*)

Exercise 6.3: From 11D SUGRA to 10D Type IIA SUGRA (*5 credits*)

The field content of eleven dimensional supergravity is the metric $g_{MN} = g_{NM}$, an antisymmetric three-tensor A_{MNR} and a gravitino Ψ_M with $\Gamma^M \Psi_M = 0$.

(a) How many degrees of freedom do these fields have? Remember that they transform under the little group. (*1 credit*)

(b) Perform the dimensional reducion to 10 dimensions on a circle. What is the massless spectrum, i.e. which representations of the ten dimensional Lorentz group $SO(9,1)$ appear from the given representations of $SO(10,1)$? Count their degrees of freedom and compare to the 11 dimensional case. (*4 credits*)