Exercises on String Theory I

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–Home Exercises–
Due 13. December 2011

Exercise 7.1: Differential Forms *12 Credits*

Totally antisymmetric lower-index tensors are an important class of tensors, called differential forms. Given such a tensor $A_{\mu_1\dots\mu_p}$, antisymmetric in all its indices, the corresponding p-form A_p is defined as

$$A_p = \frac{1}{p!}\,A_{\mu_1\dots\mu_p}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\mu_p}\,.$$

Here the wedge product of the basis one-forms is antisymmetric, $\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = -\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu$. The wedge product extends to arbitrary forms,

$$\begin{aligned} A_p\wedge B_q &= \frac{1}{p!}\frac{1}{q!}\,A_{\mu_1\dots\mu_p}B_{\nu_1\dots\nu_q}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\mu_p}\wedge\mathrm{d}x^{\nu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\nu_p}\\ &= \frac{1}{(p+q)!}\left(A_p\wedge B_q\right)_{\mu_1\dots\mu_{p+q}}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\mu_{p+q}}\,. \end{aligned}$$

Hence the components of the product form are given by (the square brackets indicate antisymmetrisation)

$$\left(A_p\wedge B_q\right)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\,A_{[\mu_1\dots\mu_p}B_{\mu_{p+1}\dots\mu_{p+q}]}\,.$$

Clearly, the degree of a form cannot exceed the spacetime dimension.
One reason for the importance of forms is that they allow for a type of derivative which does not require a connection, the exterior derivative d. It increases the degree of the form and act as follows:

$$\begin{aligned} \mathrm{d}A_p &= \mathrm{d}\left(\frac{1}{p!}A_{\mu_1\dots\mu_p}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\mu_p}\right)\\ &= \frac{1}{p!}\,\partial_\rho A_{\mu_1\dots\mu_p}\mathrm{d}x^\rho\wedge\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\cdots\mathrm{d}x^{\mu_p}\,. \end{aligned}$$

In other words, the components of the resulting $(p+1)$-form are

$$\left(\mathrm{d}A_p\right)_{\mu_1\dots\mu_{p+1}} = (p+1)\,\partial_{[\mu_1}A_{\mu_2\dots\mu_{p+1}]}\,.$$

1. Verify that the result of the exterior derivative is indeed a tensor. Furthermore, show that $\mathrm{d}^2 = 0$ and that the exterior derivative satisfies a Leibniz rule,

$$\mathrm{d}\left(A_p \wedge B_q\right) = \mathrm{d}A_p \wedge B_q + (-1)^p \, A_p \wedge \mathrm{d}B_q \, .$$

(*3 credits*)

2. How many independent components does a p-form have in d spacetime dimensions? Given a (Lorentzian) metric, we can assign to a p-form A_p a $(d-p)$-form $(*A)_{d-p}$ with components

$$(*A)_{\mu_1 \dots \mu_{d-p}} = \frac{1}{p!} \sqrt{-g} \, \varepsilon_{\mu_1 \dots \mu_d} g^{\mu_{d-p+1}\nu_1} \dots g^{\mu_d \nu_p} A_{\nu_1 \dots \nu_p}$$

Here $\varepsilon_{\mu_1 \dots \mu_d}$ is the totally antisymmetric Levi-Civita symbol, $\varepsilon_{012\dots d} = 1$, and g is the determinant of the metric. Show that this is indeed a tensor. (It suffices to show that $\sqrt{-g}\, \varepsilon_{\mu_1 \dots \mu_d}$ is a tensor, the so-called Levi-Civita tensor.) This operation is called Hodge-$*$. Compute the action of $**$. (*2 credits*)

3. Specialise to three-dimensional Euclidean space. Consider a scalar function $\phi(x)$ and a vector field $\vec{u}(x)$ and express the usual operations grad, curl and div in form language. Derive the well-known identities
   (a) $\mathrm{curl}\, \mathrm{grad}\, \phi = 0$,
   (b) $\mathrm{div}\ \mathrm{curl}\, \vec{u} = 0$,
   (c) Let $\vec{v}$ be another vector field. Express the cross product $\vec{u} \times \vec{v}$ by forms.

(*2 credits*)

4. Show that the volume form V is $V = *1$. Show further that for two p-forms A_p and B_p, we have $A \wedge *B = B \wedge *A$. (*2 credits*)

5. Consider Stokes' theorem

$$\int_V \mathrm{d}\omega = \int_{\partial V} \omega \, ,$$

where ω is a d-form and V is a $d+1$-dimensional domain. What is the meaning of this theorem for $d = 0, 1, 2$? (*3 credits*)

Exercise 7.2: Tensor scalar duality and the Stückelberg mass *8 Credits*

We first begin with a four dimensional theory of a massless two-form tensor field B_2. The action is given by

$$S = \int H_3 \wedge *H_3 \sim \int \mathrm{d}^4x \; H_{\mu\nu\rho} H^{\mu\nu\rho} \, ,$$

where $H_3 = \mathrm{d}B_2$.

1. What is the gauge symmetry which leaves the action invariant? How many degrees of freedom does B_2 have? (*2 credits*)

2. We can reparametrize the theory by regarding H_3 as fundamental field. Then we have to enforce $\mathrm{d}H_3 = 0$ using a Lagrange multiplier ϕ. Show that integrating out H_3 leads to an action for the massless scalar ϕ. What is the symmetry of ϕ? *(3 credits)*

3. We go back to the tensor theory and add a Chern–Simons coupling to a U(1) gauge theory, i.e.

$$S = \int H_3 \wedge *H_3 + cB_2 \wedge F_2 + F_2 \wedge *F_2 \tag{1}$$

with $F_2 = \mathrm{d}A_1$. Repeat the above procedure to eliminate H_3. Show that in order to make S gauge invariant, ϕ has to transform as an axion. Show that you can gauge away ϕ to obtain a massive vector boson theory. *(3 credits)*