Physikalisches Institut
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Theoretische Physik

Exercise 09
20. December 2011
WS 11/12

Exercises on String Theory I

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–Home Exercises–
Due 10. January 2012

Exercise 9.1: Green–Schwarz terms from M-Theory *20 Credits*

We compactify eleven dimensional SUGRA of a orbifold $S^1/\mathbb{Z}_2$ in the sence of Hořawa Witten, i.e. such that gauge theories at the boundaries arise. We parametrize the circle by $\phi \in [-\pi, \pi]$, i.e. $\phi \sim \phi + 2\pi$ and the orbifold acts as $\phi \mapsto -\phi$ and has two fixed points at $\phi = 0, \pi$. We are interested in the topological Chern Simons action

$$S_{\text{topo}} = -\frac{1}{12\kappa^2} \int_{M_{10}\times S^1/\mathbb{Z}_2} C \wedge G \wedge G \,, \tag{1}$$

where C is the three form fields and $G = \mathrm{d}C + \ldots$. In order to describe localisation in the eleventh dimension we define on $[-\pi, \pi]$ the forms

$$\epsilon_1(\phi) = \mathrm{sgn}(\phi) - \frac{\phi}{\pi}\,, \qquad \epsilon_2(\phi) = -\frac{\phi}{\pi}\,,$$
$$\delta_1 = \delta(\phi)\mathrm{d}\phi\,, \qquad \delta_2 = \delta(\phi - \pi)\mathrm{d}\phi\,.$$

1. Show that

   - $\mathrm{d}\epsilon_i = 2\delta_i - \frac{\mathrm{d}\phi}{\pi}$
   - $\int_{S^1} \mathrm{d}\phi\, \epsilon_i = 0$
   - $\int_{S^1} \mathrm{d}\phi\, \epsilon_i\epsilon_j = \pi\left(\delta_{ij} - \frac{1}{3}\right)$

   Show furthermore that $\delta_i\epsilon_j\epsilon_k = \frac{1}{3}\delta_{ij}\delta_{ik}\delta_k$ *Hint: Use the regularization*

   $$\epsilon_1^\eta = \begin{cases} \epsilon_1(\phi) & \phi \notin [-\eta, \eta] \\ \left(\frac{1}{\eta} - \frac{1}{\pi}\right)\phi & \phi \in [-\eta, \eta] \end{cases}\,,$$

   ϵ_2^η *similarly and* $\delta_i^\eta := \frac{1}{2}\left(d\epsilon_i^\eta + \frac{d\phi}{\pi}\right)$. (*3 credits*)

2. Show that invariance of (1) under the $\mathbb{Z}_2$ implies that C_{ABC} are odd whereas $C_{AB,11}$ are even components of C_3. $A, B, C = 0, \ldots, 9$. *Hint: What terms does* (1) *contain? How do the derivatives transform?* (*2 credits*)

3. From Hořava Witten we know that

$$\mathrm{d}G = \gamma \sum_i \delta_i \wedge I_{4,i}\,, \qquad \text{with} \quad I_{4,i} = \frac{1}{(4\pi)^2}\left(\operatorname{tr} F_i^2 - \frac{1}{2}\operatorname{tr} R^2\right)\,. \tag{2}$$

The two-dimensional descent equations read

$$I_{4,i} = \mathrm{d}\omega_i\,, \qquad \delta\omega_i = \mathrm{d}\omega_i^1\,, \tag{3}$$

where δ denotes infinitesimal gauge- and local Lorentz transformations with parameters Λ^g, Λ^L, and

$$\begin{aligned}\omega_i &= \frac{1}{(4\pi)^2}\left(\operatorname{tr}(A_i\mathrm{d}A_i + \frac{2}{3}A_i^3) - \frac{1}{2}\operatorname{tr}(\Omega_i\mathrm{d}\Omega_i + \frac{2}{3}\Omega_i^3)\right)\,,\\ \omega_i^1 &= \frac{1}{(4\pi)^2}\left(\operatorname{tr}(\Lambda^g\mathrm{d}A_i) - \frac{1}{2}\operatorname{tr}(\Lambda^L\mathrm{d}\Omega_i)\right)\,.\end{aligned}$$

The transformations act on the gauge- and spin connection as $A \mapsto (1+\Lambda^g)(A - \mathrm{d}\Lambda^g)(1-\Lambda^g)$ and $\Omega \mapsto (1+\Lambda^L)(\Omega - \mathrm{d}\Lambda^L)(1-\Lambda^L)$. The curvatures are $F = \mathrm{d}A + A\wedge A$ and $R = \mathrm{d}\Omega + \Omega\wedge\Omega$. We drop the ϕ dependence in A, F, Ω, R, Λ. Show that (3) are indeed fulfilled. *(4 credits)*

4. Show that (2) is solved by

$$G = \mathrm{d}C + (b-1)\gamma\sum_i \delta_i \wedge \omega_i + \frac{b}{2}\gamma\sum_i \epsilon_i I_{4,i} - \frac{b}{2\pi}\gamma\mathrm{d}\phi\wedge\sum_i \omega_i\,,$$

where b is a (so far) free parameter. *(2 credits)*

5. Show that invariance of G implies that C transforms as

$$\delta C = \mathrm{d}B_2^1 - \gamma\sum_i\left(\frac{b}{2}\epsilon_i\mathrm{d}\omega_i^1 + \delta_i\wedge\omega_i^1\right)$$

with some two–form B_2^1. *(2 credits)*

6. Since $C_{ABC} = 0$, it must in particular be gauge invariant. Show that this is guaranteed by $B_2^1 = \gamma\frac{b}{2}\sum_i \epsilon_i\omega_i^1$. *(2 credits)*

7. Now since G is globally well-defined, $\mathrm{d}G$ is exact and we can use Stokes theorem. First let $\mathcal{C}_5 = \mathcal{C}_4 \times S^1$ where $\mathcal{C}_4$ is a closed (= no boundary) cycle in M_{10} and S^1 is the 11^{th} dimension. Integrate $\mathrm{d}G$ over $\mathcal{C}_5$ and use (2) to show that

$$\int_{\mathcal{C}_4}\sum_i I_{4,i} = 0\,. \tag{4}$$

(2 credits)

8. Now let $\mathcal{C}_5 = \mathcal{C}_4 \times I$ where $I = [\phi_1, \phi_2]$ with $-\pi < \phi_1 < 0 < \phi_2 < \pi$. Show that now integration over $\mathcal{C}_5$ and Stokes theorem yield

$$(1-b)\int_{\mathcal{C}_4} I_{4,1} = 0\,.$$

(3 credits)