Physikalisches Institut
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Theoretische Physik

Exercise 10
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WS 11/12

Exercises on String Theory I

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–Home Exercises–
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On this exercise sheet we examine T–duality on the world–sheet. In the first exercise we consider the group $\mathrm{PSL}(2,\mathbb{Z})$ and some of its properties, which are relevant for T–duality of the torus. In the second exercise we investigate the consequences of T–duality for an open string on a circle and make a connection to D–branes.

Exercise 10.1: The group $\mathbf{PSL(2,\mathbb{Z})}$ (*10 credits*)

We define the group $\mathrm{SL}(2,\mathbb{Z})$ and its action on $z \in \mathbb{C}$ as

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}\,, \quad ad-bc=1 \right\}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}\,.$$

Furthermore, we define $\mathrm{PSL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathbb{1}\}$, and the upper half–plane $\mathfrak{H}$ of $\mathbb{C}$ as $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The aim of the exercise is to find the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ and to show that its generators can be taken to be

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,.$$

(a) Look at the order of S as a matrix and as an action on $\mathfrak{H}$. Argue why it is sensible to consider the subgroup $\mathrm{PSL}(2,\mathbb{Z})$. (*1 credit*)

(b) Define $G := \langle S,T \rangle \subseteq \mathrm{PSL}(2,\mathbb{Z})$ and let $z \in \mathfrak{H}$. Show that there exists a $g_0 \in G$ such that $\mathrm{Im}(gz) \leq \mathrm{Im}(g_0 z)$ for all $g \in G$ and z fixed. (*3 credits*)
Hint: It is easier to prove this for $g \in SL(2,\mathbb{Z})$, which implies validity for $g \in G$.

(c) Apply an S transformation to $g_0 z$ to show that $|g_0 z| \geq 1$. (*1 credit*)

(d) Repeat the above argument to show that $|T^n g_0 z| \geq 1$ for any $n \in \mathbb{Z}$. Argue furthermore that one can now use T transformations to achieve $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. What is thus the fundamental domain $\mathcal{F}$ of G? (*2 credits*)

Next, we show that S and T indeed generate $\mathrm{SL}(2,\mathbb{Z})$, i.e. $G = \mathrm{SL}(2,\mathbb{Z})$. To do so we use that every point in $\mathfrak{H}$ can be moved to $\mathcal{F}$ using an element of G. We thus choose a fixed $z \in \mathfrak{F}$, apply an arbitrary $\mathrm{SL}(2,\mathbb{Z})$ transformation γ to it and show that we can bring the result back into $\mathcal{F}$.

(e) Let $z = 2i$. Argue that $\gamma z \in \mathfrak{H}$. Thus there is a $g \in G$ such that $g(\gamma(2i)) \in \mathcal{F}$ and $g\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Using $g\gamma z \in \mathcal{F}$, calculate the values of a, b, c, d for

$$g\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to show that $\gamma = \pm g^{-1}$ and thus $\gamma \in G$. *(3 credits)*

Exercise 10.2: T–duality and D–branes ***(10 credits)***

Consider string theory on $\mathcal{M}^{1,8} \times S^1$ where $\mathcal{M}^{1,8}$ is the nine–dimensional Minkowski space. The radius of the S^1 is denoted by R. We define an operator H which maps

$$H : \begin{cases} X_L^9 & \mapsto & X_L^9 \\ X_R^9 & \mapsto & -X_R^9 \end{cases} .$$

(a) Recall the spectrum of the closed string, including Kaluza–Klein and winding modes. Is H a symmetry of the spectrum? *(4 credits)*

(b) Find the spectrum for open strings with Neumann boundary conditions along X^9, i.e. $\partial_\sigma X^9\big|_{\sigma=0,\pi} = 0$. What is the action of H on the boundary conditions and the spectrum? *(4 credits)*

(c) Repeat the exercise for Dirichlet boundary conditions, i.e. $X^9(\sigma = 0) = 0$ and $X^9(\sigma = \pi) = l$. *(2 credits)*