Exercises on Theoretical Particle Physics

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–Home Exercises–
Due 25 October 2013

H 1.1 The Lorentz group *1+2+2+1+2+3+4+4+1=20 points*

The Lorentz group O(3,1) is defined as the set of transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

which leave the scalar product $\langle x, y\rangle = \eta_{\mu\nu}\, x^\mu y^\nu$ invariant, where η is the Minkowski metric

$$\eta = \mathrm{diag}(-1, 1, 1, 1)\,.$$

(a) Show that invariance of the scalar product implies

$$\eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}\,. \tag{1}$$

(b) Use the previous equation to show that $|\Lambda^0{}_0| \geq 1$ and $|\det(\Lambda)| = 1$. Argue that this splits O(3,1) into four (disconnected) branches.

(c) Show that the subset

$$\mathrm{SO}^+(3,1) = \{\Lambda \in \mathrm{O}(3,1)\,|\, \Lambda^0{}_0 \geq 1\,,\, \det(\Lambda) = 1\}$$

forms a subgroup of SO(3,1), the so-called *proper orthochronous Lorentz group*.

(d) Identify the Lorentz transformations associated to time and parity reversal and relate them to the respective branches.

(e) Some Lorentz transformations can be written in the following exponential form $\Lambda = e^{-\mathrm{i}\lambda}$, with λ some 4×4 matrix. To which branch do these transformations belong? Explain your answer. Show that λ satisfies:

$$\lambda^T = -\eta\lambda\eta\,.$$

Hint: Reformulate the statement about the invariance of the scalar product in (1) *and write an element of the Lorentz group as* $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - i\lambda^\mu{}_\nu$.

(f) Choose

$$(M^{\mu\nu})^{\rho}{}_{\sigma} = \mathrm{i}\,(\eta^{\mu\rho}\delta^{\nu}{}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}{}_{\sigma})$$

as a basis for the Lie algebra. What do these matrices look like? Describe the form of the matrices in words. Verify the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\,(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}) \ .$$

(g) We split the generators into two groups:

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}\,, \qquad K^i = M^{0i}\,.$$

The J's have only spatial indices, the K's have spatial and timelike indices. Verify the commutation relations

$$\left[J^i, J^j\right] = \mathrm{i}\,\epsilon^{ijk}J^k\,, \qquad \left[J^i, K^j\right] = \mathrm{i}\,\epsilon^{ijk}K^k\,, \qquad \left[K^i, K^j\right] = -\mathrm{i}\,\epsilon^{ijk}J^k\,,$$

and describe the meaning of each relation in words. What kind of transformations do the J's and K's correspond to?

(h) The form of the commutation relations for the Lorentz algebra can still be simplified. Define

$$T^i_{\mathrm{L/R}} = \frac{1}{2}\left(J^i \pm \mathrm{i}\,K^i\right)$$

and verify the commutation relations

$$\left[T^i_{\mathrm{L}}, T^j_{\mathrm{L}}\right] = \mathrm{i}\,\epsilon^{ijk}\,T^k_{\mathrm{L}}\,, \qquad \left[T^i_{\mathrm{R}}, T^j_{\mathrm{R}}\right] = \mathrm{i}\,\epsilon^{ijk}\,T^k_{\mathrm{R}}\,, \qquad \left[T^i_{\mathrm{L}}, T^j_{\mathrm{R}}\right] = 0\,.$$

(i) Classify the representations of the Lorentz algebra using your knowledge of SU(2).