Physikalisches Institut
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Theoretische Physik

Exercise 9
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WS 13/14

Exercises on Theoretical Particle Physics

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Due 20 December 2013

H 9.1 Muon Lifetime *1.5+1.5+3.5+0.5+0.5+1.5+1.5+1.5+2+1 = 15 points*

The muon is the lightest unstable particle of the standard model. The goal of this exercise is to compute its lifetime. The decay rate Γ, is the decay probability per unit time, e.g.

$$\frac{dN}{N} = -\Gamma dt\,.$$

The lifetime $\tau = 1/\Gamma$, is the time it takes for a sample to decay until one is left with a fraction $1/e$ of its original ammount.

(a) Use your knowledge from the previous exercises to draw the Feynman diagram for the muon decay.

(b) In the rest frame of the muon, the gauge boson which mediates the decay is produced almost at rest. Why? Show that the corresponding matrix element is given by

$$\mathcal{M} = \frac{g^2}{8M_W^2}\left[\overline{u}(\nu_\mu)\gamma^\sigma(1-\gamma_5)u(\mu)\right]\left[\overline{u}(e)\gamma_\sigma(1-\gamma_5)v(\bar{\nu}_e)\right].$$

Hint: The propagator for a massive gauge boson is given by $-\mathrm{i}g^{\rho\sigma}(q^2-m^2)^{-1}$*, where* m *is its corresponding mass.*

(c) As the masses of the electron and the neutrinos are sufficently small compared with the muon mass we can safely neglect them in the upcoming computations.
In an analogous way as you proceeded in exercise H 7.2, compute $\langle|\mathcal{M}|^2\rangle$: average over initial spin states and sum over final ones, use the corresponding completeness relations and finally use trace identinties. The final result should read

$$\langle|\mathcal{M}|^2\rangle = 2\left[\frac{g}{M_W}\right]^4 (p_\mu\cdot p_{\bar{\nu}_e})(p_e\cdot p_{\nu_\mu})\,,$$

where the p's are the four momenta of the particles, e.g. $p_e = (E_e, \vec{p}_e)$.

(d) Make use of the kinematics to show that in the rest frame of the muon, the spin averaged matrix element takes the form

$$\langle|\mathcal{M}|^2\rangle = \left[\frac{g}{M_W}\right]^4 m_\mu^2 E_{\bar{\nu}_e}(m_\mu - 2E_{\bar{\nu}_e}).$$

(e) According to Fermi's golden rule, the rate at which decays occur can be expressed as a product of physical and kinematical factors, i.e.

$$d\Gamma = \langle|\mathcal{M}|^2\rangle \times d\Pi\,,$$

where $d\Pi$ is the so called phase space element, which dor the specific case we are considering is given by

$$d\Pi = \frac{1}{2m_\mu}\left(\frac{d^3\vec{p}_{\bar{\nu}_e}}{(2\pi)^3 2E_{\bar{\nu}_e}}\right)\left(\frac{d^3\vec{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}\right)\left(\frac{d^3\vec{p}_e}{(2\pi)^3 2E_e}\right) \times (2\pi)^4\delta^4(p_\mu - p_{\bar{\nu}_e} - p_{\nu_\mu} - p_e)\,.$$

As a first step, integrate out $\vec{p}_{\nu_\mu}$, you should obtain

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{16(2\pi)^5 m_\mu}\left(\frac{d^3\vec{p}_{\bar{\nu}_e} d^3\vec{p}_e}{E_{\bar{\nu}_e}E_{\nu_\mu}E_e}\right) \times \delta(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e)\,. \tag{1}$$

(f) Recall that we have assumed that all decay products are massless, show that in this approximation the following relation holds

$$E_{\nu_\mu}^2 = E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e}E_e\cos\theta\,,$$

where θ is the angle between $\vec{p}_e$ and $\vec{p}_{\bar{\nu}_e}$. Next write $d^3\vec{p}_e$ in spherical coordinates and aided by the previous relation, show that it can be written as

$$d^3\vec{p}_e = -\frac{E_{\bar{\nu}_e}E_{\nu_\mu}}{E_e} dE_{\bar{\nu}_e} dE_{\nu_\mu} d\phi\,,$$

with ϕ being the polar angle (i.e. $\phi \in [0, 2\pi)$).

(g) Use the previous relations to show that integrating out angular dependencies, $d\Gamma$ takes the form

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle d^3\vec{p}_e dE_{\bar{\nu}_e}}{16(2\pi)^4 m_\mu E_e^2} \int_{|E_{\bar{\nu}_e}-E_e|}^{|E_{\bar{\nu}_e}+E_e|} dE_{\nu_\mu}\delta(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e)\,. \tag{2}$$

(h) Show that, according to this expression, $d\Gamma$ vanishes unless

$$\frac{1}{2}\left(|E_{\bar{\nu}_e} - E_e| + E_{\bar{\nu}_e} + E_e\right) \leq \frac{m_\mu}{2} \leq \left(E_{\bar{\nu}_e} + E_e\right)\,. \tag{3}$$

The right handed side implies that the combined energy of the electron and the neutrino must be at least half of the muon mass. The left hand side stipulates that the most energy either of these particles can have is half of the muon rest energy.

(i) Use the previous result and the explicit form of the matrix element to integrate out the $E_{\bar{\nu}_e}$ dependence. *Hint: what are the limits of integration?*

(j) Finally integrate over $d^3\vec{p}_e$, to obtain the decay rate. Make a numerical estimation of the muon lifetime.

H 9.2 Group Theory I: The $\mathfrak{su}(N)$ Lie Algebra *2.5+2+0.5 = 5 points*

Consider the space of all $N \times N$ matrices and regard it as a Lie algebra $\mathfrak{gl}(N)$. We choose as a basis the elements e_{ab} with components $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

(a) Verify the multiplication rule and thus the commutator operation on the algebra

$$e_{ab}e_{cd} = e_{ad}\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad}.$$

In order to deal with the Lie algebra $\mathfrak{su}(N)$, what restrictions have to be made? Write down a basis for $\mathfrak{su}(N)$. What is the dimension?

(b) The **Cartan algebra** $\mathfrak{h}$ is defined to be a maximal commuting subalgebra of the Lie algebra. Its dimension is called the **rank** of the Lie algebra. Give a possible choice for the Cartan subalgebra of $\mathfrak{su}(N)$. What is the rank r of $\mathfrak{su}(N)$?

(c) Now we want to diagonalize the Cartan algebra in the adjoint representation which acts by the commutator

$$\mathrm{ad}\, h(g) = [h, g]$$

Perform a (complex) basis change of $\mathfrak{su}(N)/\mathfrak{h}$ to an eigenbasis of $\mathfrak{h}$. You should find,

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab}\,, \tag{4}$$

with $h = \sum_i \lambda_i e_{ii}$.

We can regard eq. (4) (for e_{ab} fixed) as a prescription for how to associate a number $(\lambda_a - \lambda_b)$ to each $h \in \mathfrak{h}$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b.$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra $\mathfrak{h}$. This dual space is denoted by $\mathfrak{h}^*$.