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Theoretische Physik

Exercise 10
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Exercises on Theoretical Particle Physics

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–Home Exercises–
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H 10.1 Dynkin diagram of $\mathfrak{su}(N)$ *2+2+1+2+1+3 = 11 points*

In the last exercise (H 9.2) we introduced the Cartan subalgebra of $\mathfrak{su}(N)$. There we also defined the roots of this Lie algebra.
Let $\alpha_1 \ldots \alpha_r$ be a fixed basis of roots so every element of $\mathfrak{h}^*$ can be written as $\rho = \sum_i c_i \alpha_i$. We call ρ **positive** ($\rho > 0$) if the first non-zero coefficient c_i is positive. Note, that the basis roots α_i are positive by definition. If the first non-zero coefficient c_i is negative, we call ρ negative. For $\rho, \sigma \in \mathfrak{h}^*$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$. A **simple root** is a positive root which can not be written as the sum of two positive roots.

(a) We choose a basis α_i for the root space:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1}, \quad i = 1, 2, \ldots, N-1 \,.$$

Verify that these roots are a basis and that they are positive with $\alpha_1 > \alpha_2 > \ldots > \alpha_{N-1}$. Show that these roots are simple roots.

Next, we define a structure that resembles a scalar product on the algebra. Let t_i be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra:

$$[x, [y, t_i]] = \sum_j K_{ij} t_j \,.$$

The **Killing form** is then defined as $\mathcal{K}(x, y) := \mathrm{Tr}(K)$.

(b) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h, h')$, where $h = \sum_i \lambda_i e_{ii}$, $h' = \sum_j \lambda'_j e_{jj}$.

The Killing form enables us to make a connection between the Cartan subalgebra, $\mathfrak{h}$, and its dual $\mathfrak{h}^*$: One can prove that if $\alpha \in \mathfrak{h}^*$, there exists a unique element $h_\alpha \in \mathfrak{h}$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \quad \forall h \in \mathfrak{h} \,.$$

(c) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find h_{α_i} from comparison with your result from (e).

With the help of the h_α, we are now able to define a scalar product on $\mathfrak{h}^*$:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \quad \text{where } \alpha_i, \alpha_j \in \mathfrak{h}^* .$$

(d) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} .$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root α_i, we associate a small circle and join the small circles i and j with $A_{ij}A_{ij}$ (no summation, $i \neq j$) lines.

(e) Draw the Dynkin diagram for $\mathfrak{su}(N)$.

(f) As an example, consider the Lie algebra of $\mathfrak{su}(2)$. The step operators are given by

$$J_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2) ,$$

and the Cartan subalgebra consists of the single element

$$h = J_3 = \frac{1}{2}\sigma_3 .$$

(i.1) Confirm that

$$e_{12} = J_+, \quad e_{21} = J_- \quad \text{and} \quad h = \frac{1}{2}e_{11} - \frac{1}{2}e_{22} .$$

(i.2) Calculate $\alpha_{J_\pm}(J_3)$.

(i.3) Choose $\alpha_1 = \alpha_{J_+}$ as the basis root, which is positive and simple. For $\alpha_1 \in \mathfrak{h}^*$, find the unique element $h_{\alpha_1} \in \mathfrak{h}$ such that

$$\alpha_1(h) = \mathcal{K}(h_{\alpha_1}, h) \quad \forall h \in \mathfrak{h} .$$

Hint: The solution is $h_{\alpha_1} = \frac{1}{2}J_3$.

(i.4) Calculate the Killing form $\mathcal{K}(h_{\alpha_1}, h_{\alpha_2})$ and draw the Dynkin diagram.

H 10.2 Representations of $\mathfrak{su}(N)$ *1+1+1+1+5 = 9 points*

(a) Recall the definition of the adjoint $\operatorname{ad} a(b) := [a, b]$.
Show that the adjoint is a representation of the Lie algebra

$$\operatorname{ad}\big([a, b]\big) = [\operatorname{ad} a, \operatorname{ad} b] , \quad \text{for } a, b \in \mathfrak{g} .$$

PLEASE NOTE!

♣ The bracket $[\cdot\,,\cdot]$ on the left-hand side denotes the abstract Lie-bracket, but on the right-hand side it denotes the commutator.

♣ The adjoint representation ad of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping ad : $\mathfrak{g} \to \mathrm{End}(V)$, where V is equal to the Lie algebra itself, i.e. $V = \mathfrak{g}$ This means that when we computed the Dynkin diagram of SU(N), we implicitly used the adjoint representation of SU(N):

$$\mathrm{ad}\, h(e_{ab}) = [h, e_{ab}]\,. \tag{1}$$

Furthermore, we had the eigenvalue equation

$$\mathrm{ad}\, h(e_{ab}) = \alpha_{e_{ab}}(h)\, e_{ab}\,, \tag{2}$$

which defined the roots $\alpha_{e_{ab}}$.
This eigenvalue equation can now be generalized to non-adjoint representations ρ on some vector space V. Let ϕ^i be a basis of V. We denote the representations of the elements of the Cartan subalgebra $h \in H$ by $\rho(h)$ and the representations of the step operators e_α by $\rho(e_\alpha)$. Then eq. (2) reads: $\rho(h)\,\phi^i = M^i(h)\,\phi^i$. Since the linear functions M^i act on elements $h \in H$ and give (real) numbers, they are elements of the dual space H^*. They are called **weights**. The corresponding vectors ϕ^i are called **weight vectors**. Note that roots are the weights of the adjoint representation!
You may have already gotten that simple roots α_j span H^*, so it is possible to reexpress the weights by simple roots $M^i = \sum_j c_{ij}\alpha_j$, where the coefficients c_{ij} are in general are in general non-integers. A weights M^i is called **positive**, if the first non-zero coefficients is positive. We write $M^i > M^j$, if $M^i - M^j > 0$.
A weight is called the **highest weight**, denoted by Λ, if $\Lambda > M^i\ \forall M^i \neq \Lambda$

(b) Suppose that ϕ^i is a weight vector with weight M^i. Show that $\rho(e_\alpha)\phi^i$ is a weight vector with weight $M^i + \alpha$ unless $\rho(e_\alpha)\phi^i = 0$.
Hint Use eqs. (1) and (2) and the fact that ρ is a representation. Thus it makes sense to think of the $\rho(e_\alpha)$ as raising operators and the $\rho(e_{-\alpha})$ as lowering operators.

(c) Consider now a representation ρ of SU(N). We denote the generators $\rho(t_a)$. For elements of the Cartan subalgebra, we may also write $\rho(h)$. Follow from

$$[\rho(t_a), \rho(t_b)] = \mathrm{i}\, f_{abc}\, \rho(t_c)\,,$$

that $-\rho(t_a)^*$ forms a representation, called the *complex conjugate* of ρ. We denote it by $\overline{\rho}$. ρ is said to be a real representation if it is equivalent to its complex conjugate.

(d) Show that if M^i is a weight in ρ, $-M^i$ is a weight in $\overline{\rho}$.
Hint: Use the fact that Cartan generators are hermitean and the definitions on the previous exercise sheet.

Now we are well equipped to construct the representations. For a finite dimensional representation we will find a state with highest weight Λ, which is annihilated by all positive root operators. Then we can get all states by acting with the lowering operators on it. In order to do this, we present the weights by the Dynkin labels

$$m_i := \frac{2\langle M, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} .$$

where M denotes a weight. The dynkin labels always consist of integer numbers which for a highest weight state are non-negative. It is easy to see that acting with $E_{-\alpha_i}$ corresponds to substracting the ^{i}th row of the Cartan matrix from the Dynkin label. Now you can construct all irreducible representations via the following procedure:

- ♢ start with the Dynkin label m with non-negative entries, representing the highest weight state
- ♢ if the ^{i}th entry of the Dynkin label m_i is positive, you can get m_i new states by substracting m_i times the ^{i}th row of the Cartan matrix
- ♢ repeat the last step for all new steps, for $i = 1 \dots r$
- ♢ at the end you should arrive at the lowest weight state with only non-positive entries in the Dynkin label.

(e) Construct the **5** and the **10** of $\mathfrak{su}(5)$ with the highest Dynkin labels $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. What are the higest Dynkin labels of the $\overline{\mathbf{5}}$ and the $\overline{\mathbf{10}}$? Also, construct the adjoint, the **24**, from the Dynkin label $(1, 0, 0, 1)$. How can you see that it is real?