Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises 2
24.10.2016
WS 2016/17

Exercises on Theoretical Particle Physics I

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Due 31.10.2016

3. The Lorentz group Part II

(20 credits)

(a) Elements of $\Lambda \in L^{\uparrow}{}_{+}$ can be written as

$$\Lambda^{\mu}{}_{\nu} = \left[\exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)\right]^{\mu}{}_{\nu}$$

where ω is an infinitesimal parameter. Calculate the infinitesimal variation of x^{μ} using this choice for $\Lambda^{\mu}{}_{\nu}$. Compare your result with part (e) of exercise 2 and use this to show

$$\left(M_{\rho\sigma}\right)^{\mu}{}_{\nu} = i\left(\eta_{\sigma\nu}\delta_{\rho}{}^{\mu} - \eta_{\rho\nu}\delta_{\sigma}{}^{\mu}\right).$$

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(2 credits)

(b) Show that

$$M = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & m_3 & -m_2 \\ n_2 & -m_3 & 0 & m_1 \\ n_3 & m_2 & -m_1 & 0 \end{pmatrix}$$

is the most general form of a matrix which satisfies $M^T = -\eta M \eta$. How many independent matrices do you need to construct a basis for such matrices? Write down a possible choice.

(3 credits)

(c) Use

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

to check that it is possible to write $(i, j, k = 1, \ldots, 3)$

$$J_k = \frac{1}{2}\epsilon_{kij}M_{ij}, \qquad K_i = -M_{0i}.$$

(2 credits)

(d) Use $(M_{\rho\sigma})^{\mu}{}_{\nu}$ as given in part (a) to explicitly verify the commutation relation

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right).$$

(4 credits)

(e) Use part (d) to verify

$$[K_i, K_j] = -i\epsilon_{ijk}J_k, \qquad [K_i, J_j] = i\epsilon_{ijk}K_k, \qquad [J_i, J_j] = i\epsilon_{ijk}J_k.$$

Life is easier if you use $(l, n, m = 1, \ldots, 3)$

$$\epsilon_{kln}\epsilon_{klm} = 2\delta_{nm}, \qquad \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

(4 credits)

(f) Use part (e) to show that

$$T_i^{L/R} = \frac{1}{2}\left(J_i \pm iK_i\right)$$

leads to the decoupled commutation relations

$$\left[T_i^L, T_j^L\right] = i\epsilon_{ijk}T_k^L, \qquad \left[T_i^R, T_j^R\right] = i\epsilon_{ijk}T_k^R, \qquad \left[T_i^L, T_j^R\right] = 0.$$

(2 credits)

(g) Use $(M_{\mu\nu})$ from part (c) to show that the parity operator

$$P = \mathrm{diag}\left(1, -1, -1, -1\right)$$

maps T_i^L to T_i^R. What is the physical consequence?

(3 credits)