Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises on Theoretical Particle Physics I

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4. The Lorentz group Part III (*14 credits*)

(a) Use

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \tanh\zeta, \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and express γ and $\gamma\beta$ in terms of the rapidity ζ. Use your result to define Λ as a function of ζ. What is defined by the concrete form of Λ given in this exercise?

(*2 credits*)

(b) We define

$$\Lambda = \exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right) = \exp(-i\boldsymbol{\omega}\cdot\boldsymbol{J} - i\boldsymbol{\zeta}\cdot\boldsymbol{K})$$

as partly known from exercise 3. Use the explicit form for $(M_{\mu\nu})$ as given in part (c) of exercise 3 to write down $(\omega^{\mu\nu})$ in term of ζ^i and ω^i.

(*3 credits*)

(c) Consider $\boldsymbol{\omega} = 0$, $\zeta^2 = \zeta^3 = 0$ and $\zeta^1 = -\zeta$. Show that in this case

$$\Lambda^{\boldsymbol{\omega}=0} = \exp(i\zeta K_1).$$

Show further that with $\boldsymbol{\zeta} = 0$, $\omega^1 = \omega^2 = 0$ and $\omega^3 = -\omega$ one can write

$$\Lambda^{\boldsymbol{\zeta}=0} = \exp(i\omega J_3).$$

(*2 credits*)

(d) Let us define

$$K_1 = iN_1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J_3 = iM_3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Can you relate N_1 and M_3 to part (b) of exercise 3? Calculate M_3^2 and N_1^2 and show the following relations

$$M_3^3 = -M_3, \qquad N_1^3 = N_1.$$

(*3 credits*)

(e) Show that

$$\Lambda^{\omega=0} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda^{\zeta=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for the two different cases considered in part (c) by calculating the series expansion and using the result from part (d).

(4 credits)

5. Gamma matrices Part I *(6 credits)*

(a) We define the gamma matrices γ^μ by

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i, \qquad i = 1, \ldots, 3$$

with

$$\beta^2 = (\alpha^i)^2 = \mathbb{1}, \qquad \{\beta, \alpha^i\} = \{\alpha^i, \alpha^j\} = 0, \qquad i \neq j.$$

Show that the gamma matrices fulfill the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}.$$

(2 credits)

(b) Show the following gamma matrix relations

$$\begin{aligned} &\left(\gamma^0\right)^\dagger = \gamma^0, \qquad \left(\gamma^k\right)^\dagger = -\gamma^k \\ &\left(\gamma^0\right)^2 = \mathbb{1}, \qquad \left(\gamma^k\right)^2 = -\mathbb{1} \\ &(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \end{aligned}$$

(2 credits)

(c) The lowest dimensional matrices satisfying the Clifford algebra are 4×4 matrices. The choice of these matrices is non-unique, however there exists two particularily useful representations. The Weyl or chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

and the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where σ_i are the Pauli matrices which satisfy the anti-commutation relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\,\mathbb{1}_{2\times 2}\,.$$

Verify explicitly that both these representations satisfy the defining relation of the Clifford algebra.

(2 credits)