Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises 4
7.11.2016
WS 2016/17

Exercises on Theoretical Particle Physics I

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Due 14.11.2016

6. Gamma matrices Part II

(*6 credits*)

(a) Let us define

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3.$$

Show that

$$\left(\gamma^5\right)^\dagger = \gamma^5\,, \qquad \left(\gamma^5\right)^2 = \mathbb{1}\,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0$$

without using a particular representation for the gamma matrices.

(*2 credits*)

(b) Prove the following trace theorems

$$\begin{aligned} \mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4\eta^{\mu\nu} \\ \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) \\ \mathrm{tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\right) &= 0\,, \qquad \text{for } n \text{ odd} \\ \mathrm{tr}\,\gamma^5 &= 0 \\ \mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \end{aligned}$$

independent of a concrete representation.

(*2 credits*)

(c) Show the following contraction identities

$$\begin{aligned} \gamma^\mu\gamma_\mu &= 4\cdot\mathbb{1} \\ \gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho}\,\mathbb{1} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu. \end{aligned}$$

(*2 credits*)

7. Dirac and Weyl spinors

(14 credits)

(a) Let us introduce the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that they fulfill the commutation relation $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. What does this imply for $T_i^{L/R}$ as defined in part (f) of exercise 3?

(2 credits)

(b) Show that one can write

$$\Lambda = \exp(-i(\boldsymbol{\omega} - i\boldsymbol{\zeta}) \cdot \boldsymbol{T}^L) \exp(-i(\boldsymbol{\omega} + i\boldsymbol{\zeta}) \cdot \boldsymbol{T}^R).$$

(1 credit)

(c) We call the objects which transform under the Lorentz transformations generated by $T_i^{L/R}$ as left- or right-handed Weyl spinor $\Psi_{L/R}$. Let us further define

$$\sigma^\mu = (\mathbb{1}, \sigma^i), \qquad \overline{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

and

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu), \qquad \overline{\sigma}^{\mu\nu} = \frac{i}{4}(\overline{\sigma}^\mu \sigma^\nu - \overline{\sigma}^\nu \sigma^\mu).$$

Start with

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$$

and use part (a) to show that $\Psi_{L/R}$ transform like

$$\Psi_L \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\Psi_L, \qquad \Psi_R \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\right)\Psi_R.$$

(2 credits)

(d) Let us define a Dirac spinor in the Weyl representation as

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}.$$

Show that it transforms like

$$\Psi \to \Lambda_{\frac{1}{2}}\Psi = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Psi, \qquad \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

with the help of part (c).

(2 credits)

(e) Prove the following equation

$$[\gamma^\mu, \Sigma^{\nu\sigma}] = (M^{\nu\sigma})^\mu{}_\rho \gamma^\rho.$$

(2 credits)

(f) Derive

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$$

with the known infinitesimal transformations

$$\Lambda_{\frac{1}{2}} \approx \mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}, \qquad \Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$$

and the relation from part (e).

(2 credits)

(g) Use part (b) of exercise 5 to show that

$$\Lambda_{\frac{1}{2}}^\dagger = \gamma^0\Lambda_{\frac{1}{2}}^{-1}\gamma^0$$

and thus we find for $\bar{\Psi}$ the following transformation behaviour

$$\bar{\Psi} \to \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}, \qquad \bar{\Psi} = \Psi^\dagger\gamma^0.$$

(2 credits)

(h) Calculate how the following terms Lorentz transform

$$\bar{\Psi}\Psi, \qquad \bar{\Psi}\gamma^\mu\Psi, \qquad \bar{\Psi}\Sigma^{\mu\nu}\Psi.$$

(3 credits)