Physikalisches Institut
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Theoretische Physik

Exercises 9
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# Exercises on Theoretical Particle Physics I

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Due 19.12.2016

## 15. The $\mathfrak{su}(N)$ Lie algebra *(9 credits)*

(a) Consider the space of all $N \times N$ matrices. A possible basis is given by $e_{ab}$ with components $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$. Show

$$e_{ab}e_{cd} = e_{ad}\delta_{bc}, \qquad [e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad}.$$

*(1 credit)*

(b) We are interested in the Lie algebra $\mathfrak{su}(N)$ which means we can think of $N \times N$ matrices which are traceless. We will use $e_{ab}$ with $a \neq b$ and traceless elements like

$$H_i = \sum_j \lambda_{ij} e_{jj}$$

which form a subalgebra $\mathfrak{h}$ of $\mathfrak{su}(N)$ which is called Cartan subalgebra. Its dimension is called the rank of the algebra and equals $N-1$ in our case. Show

$$[H_i, H_j] = 0.$$

We can define the so-called adjoint of $H_i$ by $\operatorname{ad} H_i(e_{ab}) = [H_i, e_{ab}]$. Show that

$$\operatorname{ad} H_i(e_{ab}) = [H_i, e_{ab}] = (\lambda_{ia} - \lambda_{ib})e_{ab}.$$

*(1 credit)*

(c) Let us define

$$\alpha_{e_{ab}}(H_i) = \lambda_{ia} - \lambda_{ib}, \qquad \alpha_j(H_i) = \lambda_{ij} - \lambda_{i\,j+1}.$$

We call $\alpha_{e_{ab}}(H_i)$ roots. Show that all $\alpha_j(H_i)$ span a basis of the root space.

*(1 credit)*

(d) Calculate the symmetric bilinear form $\mathcal{K}(H_i, H_j)$ where we define

$$\mathcal{K}(H_i, H_j) = \operatorname{tr}(\operatorname{ad}(H_i)\operatorname{ad}(H_j)).$$

*(1 credit)*

(e) It is possible to define a unique element $H_{\alpha_i}$ such that

$$\alpha_i(H_j) = \mathcal{K}(H_{\alpha_i}, H_j) \quad \forall\, j.$$

Use your result from part (d) to identify $H_{\alpha_i}$.

(*2 credits*)

(f) Derive the Cartan matrix which is defined as

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \qquad \langle \alpha_i, \alpha_j \rangle = \mathcal{K}(H_{\alpha_i}, H_{\alpha_j}).$$

(*3 credits*)

**16. More about $\mathfrak{su}(3)$** (*11 credits*)

(a) The Gell-Mann matrices are defined as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Express the step operators

$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2), \quad V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5), \quad T_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

in terms of $e_{ab}$.

(*1 credit*)

(b) $H_1 = {}^1\!/_2\lambda_3$ and $H_2 = {}^1\!/_2\lambda_8$ may be the generators of the Cartan subalgebra of $\mathfrak{su}(3)$. Express them with $e_{ab}$. Remember the definition of $H_i$ from part (b) of exercise 15 and determine the coefficients $\lambda_{ij}$.

(*1 credit*)

(c) It is possible to use $\alpha_1 = \alpha_{e_{12}}$ and $\alpha_2 = \alpha_{e_{23}}$ as basis of the root space. Use this basis to calculate the Cartan matrix with the result from part (b). Compare your result with the general form derived in part (f) of exercise 15 and check consistency.

(*1 credit*)

(d) Instead of working with the adjoint representation we now consider the so-called fundamental representation denoted by **3** in the case of $\mathfrak{su}(3)$. Let us define the so-called weight vectors

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \phi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Use $H = aH_1 + bH_2$ and the eigenvalue equation $H\phi_i = M_i\phi_i$ to determine the so-called weights $M_i$. Express $M_i$ in terms of the roots like $M_i = c_{i1}\alpha_1 + c_{i2}\alpha_2$ and derive the coefficients $c_{ij}$. We define $M_i > M_j$ by the requirement $M_i - M_j > 0$. A weight is called the highest weight $\Lambda$ if $\Lambda > M_i \quad \forall\, M_i \neq \Lambda$. Determine $\Lambda$.

(*3 credits*)

(e) Compute the Dynkin labels of a weight $M_i$ via

$$\Lambda_{ij} = 2\frac{\langle M_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

for all weights in **3**.

(*2 credits*)

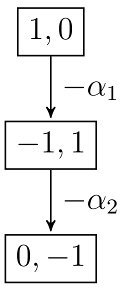
(f) It is also possible to construct all weights via the following rules

(i) Start with the weight with Dynkin labels $\Lambda_{ij}$ with non-negative entries, representing the highest weight.

(ii) For each non-negative entry $\Lambda_{ij}$ substract the $j$-th row of the Cartan matrix $\Lambda_{ij}$ times.

(iii) Repeat the last step for all weights until all Dynkin labels are non-positive.

This procedure is usually graphically depicted. In the case of the fundamental representation **3** we find




in agreement with part (e). Repeat this analysis for $(0, 1)$ which is usually denoted by $\overline{\mathbf{3}}$ and for $(1, 1)$ which is the adjoint representation.

(*2 credits*)