Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises 10
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WS 2016/17

Exercises on Theoretical Particle Physics I

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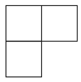
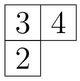
Due 9.1.2017

17. Young tableaux (*6 credits*)

(a) Young tableaux are an easy possibility to determine products of representations. A representation is depicted as a collection of boxes. Boxes in rows indicate symmetric combinations, whereas boxes in columns indicate antisymmetric combinations. There is a rule which combination of boxes corresponds to which representation. We show this rule for an example in $\mathfrak{su}(3)$

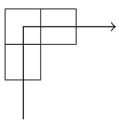
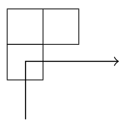
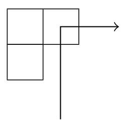
$\mathbf{8} =$  .

The dimension of the representation is given as $D = {}^{F}\!/\!_{H}$. We find F as product of all numbers in the boxes of a Young tableau where we write N in the upper left box and increasing integers to the right and decreasing integers for lower boxes. We find

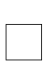
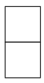
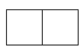
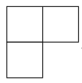


, $\qquad F = 3 \cdot 4 \cdot 2 = 24.$

To determine H we have to consider all pathes which cross the boxes from the bottom, perform a rotation to the right and leave the diagram. We have to count the number of crossed boxes as indicated for our example

 $= 3,$  $= 1,$  $= 1.$

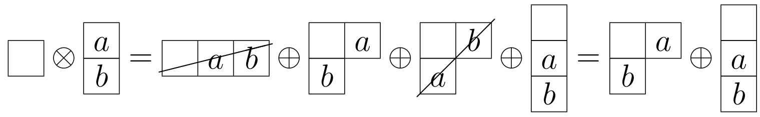
H is now the product of these numbers, which means in our case $H = 3 \cdot 1 \cdot 1$ which results in $D = {}^{24}\!/\!_{3} = 8$ which proves our example. It may also be important if the boxes are in symmetric or antisymmetric combination to extract the representation from a Young tableau. Determine now which representations belong to

, , , .

in $\mathfrak{su}(2)$ and in $\mathfrak{su}(3)$.

(*2 credits*)

(b) To build products with the help of Young tableaux one writes a's in the first row, b's in the second row and so on into the boxes of the right Young tableau in the product. Then start to add boxes from the right tableau to the left tableau row by row and take the direct sum of all possibilities. No more then N entries are allowed in one column due to antisymmetry. Collect further all a's and b's from right to left and from top to down. There should never be less a's then b's in this procedure. An example for $\mathfrak{su}(3)$ is



where we crossed out the tableaux which are not allowed. Which are the representations in this product?

(*1 credit*)

(c) Use part (b) to determine $\mathbf{2} \otimes \mathbf{2}$ and $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ for $\mathfrak{su}(2)$.

(*1 credit*)

(d) Determine $\mathbf{3} \otimes \mathbf{3}$, $\bar{\mathbf{3}} \otimes \mathbf{3}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ for $\mathfrak{su}(3)$. SU(3) is the gauge group of QCD. What is the physical consequence of the determined products?

(*2 credits*)

18. GUT breaking of SU(5) (*9 credits*)

(a) Construct all weights of the $\mathbf{5}$ of $\mathfrak{su}(5)$ from the highest weight $(1,0,0,0)$ similar to part (f) of exercise 16. The Cartan matrix has been determined in part (f) of exercise 15. Repeat the analysis for the antifundamental representation $\bar{\mathbf{5}}$ with highest weight $(0,0,0,1)$.

(*1 credit*)

(b) Repeat the analysis of part (a) for the $\mathbf{10}$ with highest weight $(0,1,0,0)$ and for the adjoint $(1,0,0,1)$ which can be labeled by $\mathbf{24}$.

(*2 credits*)

(c) It is possible to break $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. We can find how the states in a representation decompose with a simple rule. We can write the weights for the generators H_i obtained in part (a) and part (b) like $M = (M_i(H_1), M_i(H_2), M_i(H_3), M_i(H_4))$ and then break SU(5) if we discard e.g. H_3

$$M = (M_i(H_1), M_i(H_2), M_i(H_3), M_i(H_4)) \to (M_i(H_1), M_i(H_2)|M_i(H_4)).$$

$(M_i(H_1), M_i(H_2))$ is now the weight under $\mathfrak{su}(3)$ and $M_i(H_4)$ the weight under $\mathfrak{su}(2)$. Follow this rule for all states obtained in part (a) and part (b) and show with the help of part (f) from exercise 16

$$\mathbf{5} \to (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}), \quad \bar{\mathbf{5}} \to (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$$
$$\mathbf{10} \to (\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{3},\mathbf{2}), \quad \mathbf{24} \to (\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{2}) \oplus (\mathbf{3},\mathbf{2}).$$

Here the first number in brackets is the representation under $\mathfrak{su}(3)$ and the second under $\mathfrak{su}(2)$.

(*2 credits*)

(d) Use part (c) to show that all particles of one Standard Model family fit into the determined representations of SU(5) (consider only the non-Abelian gauge groups). Comment on baryon number in such a setup. Show that the gauge bosons fit into the adjoint denoted by **24**. What is the physical consequence of the other states in **24**?

(*2 credits*)

(e) We have not considered the Abelian U(1) factor so far. Write down a possible generator for the U(1) and explain your choice.

(*2 credits*)

19. Spontaneous GUT breaking (*5 credits*)

(a) Assume that we want to break the GUT symmetry with a field similar to the Standard Model Higgs mechanism. We consider a field H with symmetry $H \to -H$ which transforms in the adjoint of SU(5). Show that H can be transformed into a real diagonal traceless matrix

$$H = U H_d U^\dagger, \qquad H_d = \mathrm{diag}(h_1, h_2, h_3, h_4, h_5).$$

(*1 credit*)

(b) Use part (a) to simplify the scalar potential

$$V(H) = -m^2 \,\mathrm{tr}(H^2) + \lambda_1 (\mathrm{tr}(H^2))^2 + \lambda_2 \,\mathrm{tr}(H^4)$$

and show that at the minimum of $V(H)$ one finds

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0, \qquad a = \sum_j h_j^2.$$

What is the role of the Lagrange multiplier μ? Comment on the general form of the vacuum expectation value $\langle H_d \rangle$.

(*1 credit*)

(c) Consider the kinetic term for H

$$\mathrm{tr}\left[(D_\mu H)^\dagger (D_\mu H)\right]$$

and examine what happens if H acquires a vacuum expectation value $\langle H \rangle$. Show that $[T^a, \langle H \rangle] = 0$ is necessary to avoid massive gauge bosons. What would $[T^a, \langle H \rangle] \neq 0$ imply?

(*2 credits*)

(d) Consider the result from part (e) of exercise 18 and determine $[T^a, \langle H \rangle]$ for the Abelian generator. What is the physical consequence of your result?

(*1 credit*)