Advanced Quantum Theory (WS 24/25)

Homework no. 1 (October 7, 2024)

To be handed in by Sunday, October 13!

1 Hermitean Operators

An operator $\hat{Q}$ is hermitean, $\hat{Q} = \hat{Q}^\dagger$, if it satisfies

$$\int dx \psi_1^*(x) \hat{Q} \psi_2(x) = \int dx \left(\hat{Q} \psi_1(x) \right)^* \psi_2(x) \tag{1}$$

for all functions ψ_1, ψ_2 in the physical Hilbert space. (The integral over x may be multi–dimensional, depending on the number of degrees of freedom of the system under consideration.)

1. Show that eq.(1) implies that all eigenvalues of $\hat{Q}$ have to be real. **[2P]**
2. Show that two eigenfunctions of a hermitean operator are orthogonal if they correspond to different eigenvalues. Why does this proof not work for degenerate (i.e., equal) eigenvalues? **[3P]**
3. Show that the matrix representation $\mathbf{Q}$ of a hermitean operator $\hat{Q}$ is a hermitean matrix, i.e. $\mathbf{Q} = \mathbf{Q}^\dagger$, where the hermitean conjugate $\mathbf{A}^\dagger$ of a matrix $\mathbf{A}$ is defined via the component relation $\left(\mathbf{A}^\dagger\right)_{ij} = (\mathbf{A})^*_{ji}$. *Hint:* $(\mathbf{Q})_{ij} = \int dx \psi_i^*(x) \hat{Q} \psi_j(x) \equiv \langle i | \hat{Q} | j \rangle$, where ψ_1, ψ_j are elements of the basis of the Hilbert space. **[3P]**

2 Decomposition of a Wave Function

Any element of physical Hilbert space, i.e. any physically reasonable wave function, can be written as linear superposition of orthonormal basis states:

$$\psi(x,t) = \sum_n u_n(t) \psi_n(x)\,; \tag{2}$$

a convenient way to find a complete orthonormal basis is to find the eigenfunctions of a hermitean operator (see the previous problem); orthnormality here means

$$\int dx \psi_i^*(x) \psi_j(x) = \delta_{ij}\,, \tag{3}$$

where the Kronecker symbol $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq 0$. In this problem we will assume for simplicity that this Hilbert space has countable dimension; e.g. the ψ_n could be eigenfunctions of a hermitean operator with purely discrete spectrum of eigenvalues.

1. Using the orthonormality of the basis, show that the coefficients $u_n(t)$ can be computed from

   $$u_n(t) = \int dx \psi_n^*(x) \psi(x,t)\,. \tag{4}$$

   [2P]
2. Show that the normalization $\int dx |\psi(x,t)|^2 = 1$ implies $\sum_n |u_n(t)|^2 = 1$. **[3P]**
3. Show that the expectation value $\langle Q \rangle$ satisfies

   $$\langle Q \rangle \equiv \int dx \psi^*(x,t) \hat{Q} \psi(x,t) = \sum_n q_n \left|u_n(t)\right|^2$$

   if the ψ_n in eq.(2) are eigenfunctions of $\hat{Q}$ with eigenvalues q_n. **[3P]**

3 Angular Momentum Operator

In class we saw that the $z-$component of the angular momentum operator can be written in spherical coordinates as

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}\,, \tag{5}$$

where ϕ is the polar angle.

1. Show that the

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{im\phi} \tag{6}$$

are normalized eigenfunctions of $\hat{L}_z$ with eigenvalues $\hbar m$. **[1P]**

2. Physically the angle ϕ is the same as the angle $\phi + 2\pi$. Show that requiring $\psi_m(\phi) = \psi_m(\phi + 2\pi)$ implies that m is integer. **[2P]**

3. Show that for integer m the eigenfunctions ψ_m are indeed orthonormal, i.e. $\int_0^{2\pi} d\phi \psi_l^*(\phi)\psi_m(\phi) = \delta_{lm}$. **[2P]**

4 Canonical Transformations

In this exercise we review canonical transformations in the Hamiltonian formulation of classical mechanics, which has close formal analogies to quantum mechanics. Consider a system with N degrees of freedom, described by N generalized coordinates q_i and their canonically conjugated momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$, where $L(q_i, \dot{q}_i)$ is the Lagrange function describing the dynamics of the system. Consider a transformation of the $2N$ coordinates of phase space:

$$q_i \to \bar{q}_i(q_j, p_j)\,; \quad p_i \to \bar{p}_i(q_j, p_j)\,, \tag{7}$$

i.e. the new coordinates and new momenta are some functions of the original coordinates and momenta. Eqs.(7) define a *canonical transformation* if the following three relations for Poisson brackets hold:

$$\{\bar{q}_i, \bar{q}_k\} = \{\bar{p}_i, \bar{p}_k\} = 0\,; \quad \{\bar{q}_i, \bar{p}_k\} = \delta_{ik}\,. \tag{8}$$

The Poisson bracket is defined as $\{A, B\} \equiv \sum_j \left(\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j} \right)$.

1. Show that canonical transformations leave the Hamilton equations of motion form–invariant, i.e. one has

$$\dot{\bar{q}}_i = \frac{\partial H}{\partial \bar{p}_i}\,; \quad \dot{\bar{p}}_i = -\frac{\partial H}{\partial \bar{q}_i}\,.$$

Hint: Use the chain rule to express the derivatives of H with respect to the $\bar{q}_i$, $\bar{p}_i$ in terms of derivatives of H w.r.t. the original q_i, p_i. **[4P]**

2. Show that

$$\bar{q} = \ln(q^{-1} \sin p)\,, \quad \bar{p} = q \cot p$$

is a canonical transformation. **[2P]**

3. Show that canonical transformations also leave the Poisson brackets between arbitrary functions of the coordinates and momenta unchanged,

$$\{A(q,p), B(q,p)\}_{q,p} = \{A(\bar{q},\bar{p}), B(\bar{q},\bar{p})\}_{\bar{q},\bar{p}}\,.$$

Here the indices on the coordinates and momenta have been suppressed for simplicity, and on the right–hand side, the Poisson bracket is defined via derivatives w.r.t. the transformed quantities, as indicated by the subscript. **[4P]**