Advanced Quantum Theory (WS 24/25)

Homework no. 12 (January 8, 2024)

Please hand in your solution by Sunday, January 12!

1 Lorentz Group

The Lorentz group can be defined as the set of all rank–2 tensors Λ that leave the Minkowski metric g invariant,

$$\Lambda^T g \Lambda = g\,. \tag{1}$$

A proper Lorentz transformation in addition satisfies $\det(\Lambda) = 1$. It can be written as a product of *boosts* and *rotations*. A rotation only affects the spatial components of a 4−vector; it has an orthonormal 3×3 matrix $\mathcal{O}$ in the lower–right corner of Λ, with $\Lambda^0_0 = 1$ and $\Lambda^i_0 = \Lambda^0_i = 0$, see eq.(7.5) in class. A boost in x−direction is described by

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

see eq.(7.7) in class; here $\beta = v/c$, v being the relative velocity between the two inertial frames one is considering and c the speed of light in vacuum, and $\gamma = 1/\sqrt{1-\beta^2}$. In this exercise we investigate the properties of this group.

1. First, show that the Λ satisfying eq.(1) indeed form a group; it is called $SO(3,1)$. **[4P]**.
2. Show that the set of all rotations form a subgroup of the Lorentz group; it is called $SO(3)$. **[3P]**
3. Show that the set of all rotations around the x axis (i.e. in the (y, z) plane) form a subgroup of the group of all rotations, and hence also a subgroup of the Lorentz group. (This is true for rotations around any fixed axis, of course.) What is the single rotation that results from successive rotations by angles α_1 and α_2? **[2P]**
4. Show that the boosts along the x−axis also form a subgroup of the Lorentz group. (Again this is true for boosts along any fixed direction.) What is the single boost that describes successive boosts by β_1 and β_2? (This is also known as the relativistic addition theorem of velocities.) **[3P]**
5. Finally, show that the set of *all* boosts does *not* form a subgroup of the Lorentz group. To that end, consider successive boosts in x and y directions, by β_x and β_y, respectively. Does the order of these boosts matter? In order to show that the resulting product cannot be represented by a single boost in the (x, y)−plane (the z direction evidently doesn't play a role here, and can be neglected), write down the Λ corresponding to a boost in a general direction defined by the unit vector $\vec{n} = (\cos\theta,\ \sin\theta)$. *Hint:* Write the spatial 3−vector $\vec{x}$ as a term $\propto \vec{n}$ and a second term $\propto \vec{n}_T$, where $\vec{n}_T$ is orthogonal to $\vec{n}$; a boost in $\vec{n}$ direction only affects the component of $\vec{x}$ parallel to $\vec{n}$. **[5P]**

2 Reminder: 4–Vectors and Relativistic Kinematics

A contravariant 4−vector a^μ changes under a Lorentz transformation as

$$a^\mu \to \Lambda^\mu_\nu a^\nu \, ; \tag{3}$$

here Einstein's summation convention has been used, i.e. a sum over the repeated index ν is implied; by convention this sum runs from 0 to 3. The covariant version of this vector is defined as

$$a_\mu = g_{\mu\nu} a^\nu \, , \tag{4}$$

where $g_{\mu\nu}$ is the Minkowski metric, given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1) \, . \tag{5}$$

1. Show by explicit calculation that the scalar product of two 4−vectors a^μ and b^μ, defined via
$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} \tag{6}$$
is invariant under a boost in y−direction. **[1P]**

2. Show that $f(\vec{x}, t) = \mathrm{e}^{ip\cdot x/\hbar}$ behaves like a plane wave; hence $k^\mu = p^\mu/\hbar$ is a wave 4−vector. **[1P]**

3. Consider the two–body decay of a particle with mass M and 4−momentum p^μ into two particles with masses m_1, m_2 and 4−momenta k_1^μ, k_2^μ. Energy and momentum conservation implies
$$p^\mu = k_1^\mu + k_2^\mu \, . \tag{7}$$
Show that this is possible if, and only if, $m_1 + m_2 \leq M$. *Hint:* The possibility of this decay occurring must be independent of the reference frame, hence you can analyse the problem in a convenient frame. **[2P]**

4. Specifically, consider the case of a spinless (scalar) particle into two massless particles; an example is the decay of a neutral pion into two photons, $\pi^0(p) \to \gamma(k_1)\gamma(k_2)$. Compute the spectrum of one decay photon, $d\Gamma/dE_1$, in a frame where the pion has 4−momentum p^μ. *Hint:* First show that in the rest frame of the decaying pion, $d\Gamma/dE_1^* \propto \delta(E_1^* - Mc^2/2)$, where M is the mass of the neutral pion and the $*$ refers to the pion rest frame. Using the fact that the decay is isotropic in the rest frame, $d\Gamma/d\Omega^* = \text{const.}$, perform the boost into the frame where the pion has 4−momentum p^μ for a fixed angle θ^* between the photon direction in the pion rest frame and the flight direction of the pion, and then integrate over $\cos\theta^*$ to derive the answer. **[5P]**