Advanced Quantum Theory (WS 24/25)
Homework no. 14 (January 20, 2025): the last one!
Please send in your solution by Monday, January 27!

Quickies

Q1: Write down the Klein–Gordon equation for a free particle with mass m. **[1P]**

Q2: *(i)* What (anti–)commutation relations do the Dirac matrices α_k and β have to satisfy? *(ii)* Why do these matrices have to be hermitean? *(iii)* Why do we need three α_k matrices? **[3P]**

Q3: What is the (conserved) probability 4−current in the Dirac theory? **[1P]**

1 Lorentz Transformations and the Dirac Equation

In class we had considered infinitesimal Lorentz transformations,

$$\Lambda^{\nu}{}_{\mu} = g^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu} \tag{1}$$

with

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \tag{2}$$

(*only* for two upper, or two lower, indices!), such that a 4−vector x transforms like

$$x \to x' = \Lambda x\,. \tag{3}$$

We wanted to find the corresponding transformation of the Dirac spinor $\psi(x)$, described by

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)\,, \tag{4}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the components of ψ. For an infinitesimal transformation we made the ansatz $S = \mathbb{1}_{4\times 4} + \tau$, where τ is an infinitesimal 4×4 matrix which (of course) also acts on Dirac (spinor) indices. By demanding that $\psi'(x')$ satisfies the Dirac equation in the new frame we derived the first condition on τ:

$$[\gamma^{\alpha}, \tau] = \Delta\omega^{\alpha}{}_{\beta}\gamma^{\beta}\,. \tag{5}$$

Moreover, we normalized S such that $\det(S) = 1$, which implies that the trace of τ vanishes,

$$\mathrm{tr}(\tau) = 0\,; \tag{6}$$

recall that the trace of a matrix is the sum of its diagonal elements. Note that the determinant and trace refer only to the Dirac indices; τ does not have a free Lorentz index.

Show that the ansatz

$$\tau = -\frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} = \frac{1}{8}\Delta\omega^{\mu\nu}[\gamma_{\mu}, \gamma_{\nu}] \tag{7}$$

satisfies conditions (5) and (6). *Hint:* Prove and use that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two matrices A and B. **[6P]**

2 Dirac vs Weyl Representation

In this problem we look at a different representation of the Dirac matrices, which is particularly useful at high energies; and we'll explicitly construct a (unitary) transformation matrix that relates the two representations.

1. In class we had seen that a transformation of the type

$$\alpha_k \to U \alpha_k U^{-1}\,,\ \beta \to U \beta U^{-1} \tag{8}$$

leaves the anti–commutation relations between these four matrices invariant, for any non–singular 4×4 matrix U. Show that this transformation implies $\gamma^\mu \to U \gamma^\mu U^{-1}$, where $\gamma^0 = \beta,\ \gamma^k = \beta \alpha^k$, and show that the tranformed γ matrices satisfy the same anti–commutation relations as the original γ^μ do. **[2P]**

2. The explicit form of the γ matrices in the Dirac representation is:

$$\text{Dirac rep.}:\ \gamma_D^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\,, \tag{9}$$

where σ_k are the Pauli matrices. In the chiral or Weyl representation one has instead:

$$\text{Chiral rep.}:\ \gamma_C^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\,, \tag{10}$$

i.e. only γ^0 changes while the γ^k remain the same. We wish to find a unitary (!) 4×4 matrix U that connects these two representations. As a first step, show that

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} u_1 & -u_2 \\ u_1 & u_2 \end{pmatrix} \tag{11}$$

is indeed unitary, i.e. $UU^\dagger = U^\dagger U = \mathbb{1}$ (where $\mathbb{1}$ is the 4×4 unit matrix), if u_1 and u_2 are (in general different) unitary 2×2 matrices. **[2P]**

3. Next, show that $\gamma_C^0 = U \gamma_D^0 U^\dagger$, for arbitrary (but unitary!) u_1 and u_2. **[1P]**

4. Show that $U \gamma^K U^\dagger = \gamma^k$ (as required to keep the γ^k unchanged) is satisfied if

$$u_2 \sigma_k u_1^\dagger = \sigma_k\ \forall k\,. \tag{12}$$

Hint: Show that eq.(12) is equivalent to $u_1 \sigma_k u_2^\dagger = \sigma_k\ \forall k$. **[2P]**

5. In order to find explicit expressions for u_1 and u_2 that solve eqs.(12), show that the ansatz

$$u_l = \begin{pmatrix} \cos\theta_l & \sin\theta_l \mathrm{e}^{i\phi_l} \\ -\sin\theta_l \mathrm{e}^{-i\phi_l} & \cos\theta_l \end{pmatrix}\,,\ l = 1,\ 2 \tag{13}$$

produces unitary 2×2 matrices, i.e. $u_l u_l^\dagger = u_l^\dagger u_l = 1$. (Note that u_l has two real parameters, whereas an orthogonal 2×2 matrix, which describes rotations in a plane, has only one free parameter.) **[2P]**

6. Now let us look at eq.(12) for $k = 3$. Show that it implies (almost uniquely)

$$\theta_1 = \theta_2 \quad \text{and} \quad \phi_1 = \phi_2 + \pi \,. \tag{14}$$

7. Next consider eq.(12) for $k = 1$. After using eq.(14), show that we need

$$\phi_1 = 0 \quad \text{or} \quad \phi_1 = \pi \,. \tag{15}$$

[2P]

8. Finally, consider eq.(12) for $k = 2$. Using eqs.(14) and (15), show that it requires

$$\theta_1 = 0 \quad \text{or} \quad \theta_1 = \pi \,. \tag{16}$$

What is therefore the final form of U? **[3P]**

3 Bonus Question

This is the last tutorial of this class. Go through your notes and ask your tutor to clarify one issue for you. **[4P]**[1]

[1] When computing whether you satisfy the "50% rule", i.e. are permitted to take the final exam, these bonus points count in the numerator, but not in the denominator.