Advanced Quantum Theory (WS 24/25)
Homework no. 3 (October 21, 2024)
To be handed in by Sunday, October 27!

1 Particle in External Electromagnetic Field

In classical mechanics the Lagrange function describing the interaction of a point particle with mass m and charge q with given electromagnetic fields is given by (using Cartesian coordinates):

$$L = \frac{1}{2} m \left(\dot{\vec{x}} \right)^2 - q \left(V - \dot{\vec{x}} \cdot \vec{A} \right) ; \tag{1}$$

here V is the scalar potential and $\vec{A}$ is the vector potential.

1. Determine the canonical momentum $\vec{P}$ associated to $\vec{x}$. How is it related to the linear momentum $\vec{p}$? **[2P]**
2. Show that the corresponding Hamilton function can be written as

   $$H = \frac{1}{2m} \left(\vec{P} - q\vec{A} \right)^2 + qV \,. \tag{2}$$

   Is this equal to the total energy of the particle? **[3P]**
3. Show that the Hamiltonian equation of motion $\dot{\vec{P}} = \left\{ \vec{P}, H \right\}$ reproduces the equation of motion according to the Lorentz force,

   $$\dot{\vec{p}} = q \left[-\vec{\nabla} V - \partial \vec{A} / \partial t + \dot{\vec{x}} \times \left(\vec{\nabla} \times \vec{A} \right) \right] . \tag{3}$$

   Hint: use the identity

   $$\dot{\vec{x}} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left(\dot{\vec{x}} \cdot \vec{A} \right) - \left(\dot{\vec{x}} \cdot \vec{\nabla} \right) \vec{A},$$

   and prove and use the relation

   $$\dot{\vec{A}} = d\vec{A}/dt = \partial \vec{A} / \partial t + \left(\dot{\vec{x}} \cdot \vec{\nabla} \right) \vec{A} \,.$$

   [6P]
4. Use the appropriate Poisson brackets to argue that in quantum mechanics, $\hat{\vec{P}}$, and not the operator of linear momentum $\hat{\vec{p}}$, is represented by $-i\hbar\vec{\nabla}$. **[2P]**

2 Charge Conservation

In classical electrodynamics the conservation of electric charge is equivalent to the continuity equation relating the charge density ρ to the current density $\vec{j}$,

$$\partial \rho / \partial t + \vec{\nabla} \cdot \vec{j} = 0 \,. \tag{4}$$

Here we wish to analyze this continuity equation in the context of non–relativistic quantum mechanics.

1. The charge density is quite obviously given by

$$\rho(\vec{x},t) = q\,|\psi(\vec{x},t)|^2 \,, \tag{5}$$

where q is the electric charge of the particle. Use the results from the first problem of this sheet to argue that the current density is given by

$$\vec{j}(\vec{x},t) = \frac{q}{2m}\left[\psi^*(\vec{x},t)\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t)\right)\psi(\vec{x},t) + h.c.\right] \,, \tag{6}$$

where *h.c.* stands for the hermitean conjugate of the first term. **[2P]**

2. Show that ρ defined in (5) and $\vec{j}$ defined in (6) satisfy the continuity equation (4). *Hint:* Use the Schrödinger equation! **[4P]**

3. Show that ρ and $\vec{j}$ are invariant under a gauge transformation, where simultaneously [see (2.44) in class]:

$$\vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t) + \nabla\lambda(\vec{x},t) \quad \text{and} \quad \psi(\vec{x},t) \to \exp\left(i\frac{q}{\hbar}\lambda(\vec{x},t)\right)\psi(\vec{x},t)\,. \tag{7}$$

[3P]

3 Some Gaussian Integrals

In this exercise we compute some definite Gaussian integrals, allowing for complex parameters.

1. Show by finding the primitive of the integrand that

$$I_1(a) = \int_0^\infty dx\, x\mathrm{e}^{-ax^2} = \frac{1}{2a} \,, \tag{8}$$

where a is a complex constant with $\Re e(a) \geq 0$. (Strictly speaking the result holds only for $\Re e(a) > 0$, but it can be extended to $\Re e(a) = 0$, i.e. purely complex a. What goes wrong if $\Re e(a) < 0$?) **[2P]**

2. Now consider the seemingly simpler integral

$$I_0(a) = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ax^2} \,. \tag{9}$$

Here the primitive of the integrand cannot be expressed as an elementary function. Consider instead

$$[I_0(a)]^2 = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ax^2} \int_{-\infty}^\infty dy\, \mathrm{e}^{-ay^2} \,.$$

Hint: Use polar coordinates, $x = r\cos\phi$, $y = r\sin\phi$; the integral over r can then be reduced to I_1 of (8). **[3P]**

3. Compute

$$I_2(a) = \int_{-\infty}^\infty dx\, x^2\mathrm{e}^{-ax^2} \,,$$

by taking an appropriate derivative of $I_0(a)$. **[2P]**

4. Finally, show that

$$I_0(a,b) = \int_{-\infty}^\infty dx\, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/(4a)}\sqrt{\frac{\pi}{a}}$$

where a, b are complex constants with $\Re e(a) \geq 0$. *Hint:* Complete the square in the exponent, and use the result for $I_0(a)$! **[3P]**