Advanced Quantum Theory (WS 24/25)

Homework no. 5 (November 4, 2024)

To be handed in by Sunday, November 11!

Quickies

Q1: How are *active* and *passive* transformations defined for a quantum system? **[2P]**

Q2: Which operator generates a rotation in the (x, z) plane by angle α? **[1P]**

Q3: How does the wave function of *(i)* a proton; *(ii)* a neutron change under a gauge transformation of the electromagnetic potentials, described by the gauge function $\lambda(x, t)$? **[2P]**

1) Propagator of the Harmonic Oscillator

In this exercise we want to compute the spatial dependence of the propagator of the harmonic oscillator using the path integral formalism. For simplicity we work in one dimension, so that the Lagrangian is given by

$$L = \frac{1}{2} m \left(\dot{x}^2 - \omega^2 x^2\right) , \tag{1}$$

where m is the mass of the particle and ω the characteristic frequency of the oscillator.

1. Show that the classical action for going from point x at time $t = 0$ to point x' at time t is given by
$$S_{\rm cl}(x, x', t) = \frac{m\omega}{2\sin(\omega t)} \left[\cos(\omega t)\left(x^2 + x'^2\right) - 2xx'\right] . \tag{2}$$
Hint: Write the classical trajectory as $x_{\rm cl}(t') = a\cos(\omega t') + b\sin(\omega t')$, which allows to easily evaluate the action integral. Use the boundary conditions to replace a and b by x and x'. **[5P]**

2. Show that
$$U(x, x', t) = \int_{x'}^{x} \mathcal{D}x(t') \exp\left[\frac{i}{\hbar}\int_0^t dt'\, L(x(t'), \dot{x}(t'))\right] = F(t) \cdot \exp\left[\frac{i}{\hbar} S_{\rm cl}\right] , \tag{3}$$
where $S_{\rm cl}$ is the classical action of the first part of this problem, and $F(t)$ is some function of time only. Eq.(3) shows that it is possible to compute the shape (i.e., $x-$dependence) of the wave function $\psi(x, t)$ from the initial condition $\psi(x, 0)$ using only the classical action, i.e. no actual integration over non–classical paths is necessary; the latter only affect the normalization of the wave function at times $t \neq 0$. *Hint:* Parameterize the path as $x(t') = x_{\rm cl}(t') + y(t')$; since the beginning and the end of the path are fixed, $y(0) = y(t) = 0$. Insert this into the action integral, and show that the terms linear in y and $\dot{y}$ cancel. **[4P]**

2) Ordinary Path Integral from Phase Space Path Integral

In class we had derived the relation [see eq.(3.37)]

$$\begin{aligned} U(x, x', t) &= \int \mathcal{D}p\mathcal{D}x \exp\left[\frac{i}{\hbar}\int_0^t dt'\, \left(p\dot{x} - H(x, p)\right)\right] \\ &= \lim_{N\to\infty} \prod_{k=1}^{N-1} \int dx_k \prod_{k=1}^{N} \int \frac{dp_k}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{\epsilon p_k^2}{2m} - p_k(x_k - x_{k-1}) + \epsilon V(x_{k-1})\right)\right] \end{aligned} \tag{4}$$

where $x_0 = x'$ and $x_N = x$, with $N\epsilon = t$. Show by explicitly performing the (Gaussian) integrals over the p_k that this reduces to the expression (3.20) given in class for the usual path integral,

$$U(x,x',t) = \frac{1}{B} \lim_{N\to\infty} \prod_{k=1}^{N-1} \int \frac{dx_k}{B} \exp\left[\frac{i}{\hbar} \sum_{k=1}^{N} \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\epsilon} - \epsilon V(x_{k-1}) \right)\right], \tag{5}$$

where

$$B = \sqrt{\frac{2\pi\hbar i\epsilon}{m}}.$$

Hint: Note that the integrals over the p_k are all independent of each other. **[5P]**

3) Path Integral with Vector Potential

In this exercise we want to show the equivalence between the path integral treatment and the Schrödinger equation for the case where a point particle interacts with the vector potential of electrodynamics. That is, we want to show that

$$\psi(x,\epsilon) = \psi(x,0) - i\frac{\epsilon}{\hbar}\hat{H}\psi(x,0), \tag{6}$$

see eq.(3.21) in class, can be derived from the path integral formalism where for the case at hand

$$\psi(x,\epsilon) = \frac{1}{B}\int_{-\infty}^{\infty} d\eta\, \psi(x+\eta,0) \exp\left[\frac{i}{\hbar}\left(\frac{m\eta^2}{2\epsilon} - q\epsilon\frac{\eta}{\epsilon}A(x+\alpha\eta,0)\right)\right]. \tag{7}$$

Here the constant B is as in eq.(5), and A is the one–dimensional "vector potential". We argued in class that in order to compute to first order in ϵ, we need to expand up to *second* order in η. $-\eta/\epsilon$ becomes the velocity $\dot{x}$ in the continuum limit. Observe that the interaction term in (7) is now $\mathcal{O}(\eta)$, i.e. $\mathcal{O}(\sqrt{\epsilon})$. Show that the final answer then depends on where exactly the vector potential is taken; in case of the scalar potential any argument between x' and x (i.e. any $\alpha \in [0,1]$) was ok to first order in ϵ. Show that the equivalence between Schrödinger equation and path integral only works for the choice

$$\alpha = \frac{1}{2},$$

i.e. for the so–called *mid–point prescription. Hint:* Follow the derivation shown in class, i.e. expand $\psi(x+\eta,0)$ and $A(x+\alpha\eta,0)$ to the required order in η, perform the Gaussian integration over η, and expand the result to first order in ϵ. **[8P]**