Advanced Quantum Theory (WS 24/25)

Homework no. 7 (November 18, 2024)

To be handed in by Sunday, November 24!

Quickies

Q1: *(i)* How are "sudden" and "adiabatic" perturbations defined? *(ii)* Why can there, strictly speaking, be no adiabatic perturbation if the unperturbed system has degenerate eigenstates? **[2P]**

Q2: *(i)* How are operators in the Schrödinger and interaction pictures related? *(ii)* Show that the unperturbed, time–independent Hamiltonian $\hat{H}_0$ is the same in the Schrödinger and interaction pictures, $\hat{H}_{0S} = \hat{H}_{0I}$. **[2P]**

Q3: How does the transition rate for (stimulated) emission off an excited atomic system scale with the number of photons N_γ in the initial state? **[1P]**

1) Transitions between General States

In this problem we will analyze the probability for time–dependent transitions between states that are not (necessarily) eigenstates of the unperturbed Hamiltonian, but are eigenstates of some hermitean operator $\hat{O}$, i.e.

$$\hat{O}|\alpha\rangle = o_\alpha|\alpha\rangle \qquad \hat{O}|\beta\rangle = o_\beta|\beta\rangle \,. \tag{1}$$

We are interested in the situation where at time $t = t_0$, the system was in state $|\alpha\rangle$, and want to compute the probability that the system is in state $|\beta\rangle$ at time $t > t_0$. As in class, we assume that the total Hamiltonian is the sum of a time–independent term $\hat{H}_0$ and a time–dependent perturbation $\hat{H}_1(t)$.

1. In order to make contact with the results derived in class, express the states $|\alpha\rangle$ and $|\beta\rangle$ as linear superpositions of eigenstates $|n^{(0)}\rangle$ of $\hat{H}_0$. What does the orthogonality $\langle\beta|\alpha\rangle = \delta_{\alpha\beta}$ imply for the expansion coefficients? **[2P]**

2. Let us first consider the Schrödinger picture. Show that the probability that $|\psi_S(t)\rangle = |\beta\rangle$ is in general non–zero for $t > t_0$ even if $\hat{H}_1 = 0$, i.e. without perturbation, if the commutator $[\hat{H}_0, \hat{O}_S] \neq 0$. In this case the zeroth–order estimate for the transition rate will generally already be a good approximation, if $\hat{H}_1$ is indeed a small perturbation. **[2P]**

3. Now consider the interaction picture. Show that the probability of the interaction picture state vector $|\psi_I(t)\rangle = |\beta\rangle$ is zero for $\hat{H}_1 = 0$ and $\alpha \neq \beta$. The discrepancy to the result of the previous subsection clearly shows that the wave function, or state vector, by itself has no direct physical meaning, even though one of the axioms of quantum mechanics states that it contains all the information about the system that can be known! **[2P]**

4. Statements about the probability of the outcome of some physical measurement ought to be the same in the two pictures. Show that, even allowing for nonvanishing $\hat{H}_1$, the probability that a measurement of $\hat{O}$ at time t yields o_β, given that a measurement of $\hat{O}$ at time t_0 yielded o_α, can be written as

$$P_{\beta\alpha} = \left| \sum_{i,f} \langle i^{(0)}|\alpha\rangle\langle\beta|f^{(0)}\rangle \tilde{\mathcal{A}}_{fi}(t, t_0) \right|^2 , \tag{2}$$

where the $\tilde{A}_{fi}$ are the Schrödinger picture transition matrix elements introduced in class, $\hat{A}_{fi} = \langle f^{(0)}|\hat{U}_S(t, t_0)|i^{(0)}\rangle$. *Hint:* Compute the expectation value $\langle\hat{O}\rangle$ at time t in the

Schrödinger picture. Use the fact that $|\psi_S(t)\rangle = \hat{U}_S(t,t_0)|\alpha\rangle$, insert two complete sums over eigenstates of $\hat{O}$, use the fact that these eigenstates are orthogonal, and interpret the result expressing the expectation value as sum over eigenvalues times the probabilities we wish to compute. **[5P]**

2) Selection Rules

Here we want to consider selection rules for transitions between eigenstates of the unperturbed Hamiltonian. To that end, consider a hermitean operator $\hat{O}$ that commutes with $\hat{H}_0$, $[\hat{H}_0, \hat{O}] = 0$. This means that the eigenstates of $\hat{H}_0$ can also be assigned a well–defined eigenvalue of $\hat{O}$, i.e. we can write $|\psi(t_0)\rangle = |i^{(0)}, o_i\rangle$ and so on.

1. Assume that $[\hat{H}_1, \hat{O}] = 0$, i.e. $\hat{O}$ also commutes with the perturbation of the Hamiltonian. Show that this immediately implies

$$\langle f^{(0)}, o_f|\hat{H}_1|i^{(0)}, o_i\rangle = 0 \quad \text{if } o_f \neq o_i \,, \tag{3}$$

   i.e. first–order transitions between states with different O quantum number are forbidden. **[2P]**

2. Show that eq.(3) in fact forbids transitions between states with different O quantum number to *all orders* in perturbation theory. **[2P]**

3. Assume that $[\hat{H}_0, \hat{\vec{L}}] = 0$, where $\hat{\vec{L}}$ is the angular momentum operator. What angular momentum selection rules hold if *(i)* $\hat{H}_1 = f(\hat{z}, t)$, *(ii)* $\hat{H}_1 = g(|\hat{\vec{x}}|, t)$, for arbitrary real functions f and g? **[3P]**

3) Atomic Radiative Transitions

In this problem we consider radiative transitions in a hydrogen–like atom.

1. We saw in class that the leading perturbation of the Hamiltonian is proportional to $\vec{\epsilon} \cdot \vec{x}_e$, where $\vec{\epsilon}$ is the polarization vector of the emitted photon, and $\vec{x}_e$ the coordinate vector of the electron. Show that this can be written as

$$\vec{\epsilon} \cdot \vec{x}_e = r_e \sum_m c_m Y_{1m}(\theta_e, \phi_e) \,, \tag{4}$$

   where r_e, θ_e, ϕ_e are the spherical coordinates of the electron and the c_m are some (possibly complex) coefficients that linearly depend on the components of $\vec{\epsilon}$. **[3P]**

2. In class we considered transitions due to the term $\propto \vec{A}_{\rm rad} \cdot \hat{\vec{P}}$ in the Hamiltonian. Show that this perturbation cannot lead to transitions between different spin states of the electron, whereas the term $\propto \hat{\vec{\mu}}_e \cdot \vec{B}$, which also appears in the electromagnetic Hamiltonian of electrons with spin, can lead to such transitions; here $\vec{\mu}_e$ is the magnetic moment of the electron. *Note:* Transitions of this kind lead to the famous "21 cm radiation", used to track neutral hydrogen in the Universe with radio telescopes. **[3P]**

3. In class we had seen that the ugly $\delta-$function ensuring energy conservation in Fermi's Golden Rule disappears after integrating over the phase space of the emitted photon. Of course, this only works for emission processes. How can this $\delta-$function be removed for absorption, in the (often realistic) case where the atom is subject to a beam of parallel photons with slightly different energies? *Hint:* Replace the photon number N_γ by a spectrum $I_\gamma(\omega)$, so that $\int I_\gamma(\omega) d\omega = N_\gamma$. **[3P]**

4. Show that the second order (in e) contribution to the electromagnetic Hamiltonian, $\hat{H}_2 = \frac{e^2}{2M_E}\vec{A}\cdot\vec{A}$, see eq.(4.34) in class, allows $(2S) \to (1S)$ transitions, but does *not* allow $(2P) \to (1S)$ transitions. **[3P]**

5. Use selection rules, and the fact that the rate for spontaneous electric dipole transitions is $\propto \omega_{if}^3$, to argue that so–called Rydberg states, where both the principal quantum number n and the angular momentum quantum number l is large, have quite long lifetimes. **[3P]**