# Advanced Quantum Theory (WS 24/25)

Homework no. 10 (December 2, 2024)

**Please hand in your solution by Sunday, December 8!**

## Quickies

**Q1:** *(i)* Write down the scattering amplitude $f_{\vec{k}}$ for scattering on a fixed potential in Born approximation. *(ii)* Argue that the result does not depend on variations of the scattering potential on length scales that are much less than $1/|\vec{q}|$, where $\hbar\vec{q} = \hbar\left(\vec{k}' - \vec{k}\right)$ is the momentum exchanged in the scattering. **[3P]**

**Q2:** Assuming repulsive interactions, give two arguments why one needs high energy in order to show that the scattering center in Rutherford scattering is "pointlike". *Hint:* You may argue (semi–)classically. **[3P]**

**Q3:** *(i)* What is the unitarity limit on a partial–wave reduced scattering amplitude $a_\ell(|\vec{k}|)$? *(ii)* What is the optical theorem? **[2P]**

## 1) Scattering on a Dipole

In this problem we will compute the Coulomb scattering cross section for scattering on a physical dipole, consisting of two point particles with opposite charges $\pm Z_1 e$, one located at the origin ($\vec{x} = \vec{0}$), the other at $\vec{x} = \vec{d}$. The incident particles have charge $Z_2 e$.

1. Show that in Born approximation the scattering amplitude $f_{\vec{k}}(\theta, \phi)$ can be written as
$$f_{\vec{k}}^{\text{dipole}} = \left(1 - \mathrm{e}^{-i\vec{q}\cdot\vec{d}}\right) f_{\vec{k}}^{\text{monopole}}, \tag{1}$$
where $f_{\vec{k}}^{\text{monopole}}$ is the scattering amplitude for scattering on a single pointlike charge derived in class,
$$f_{\vec{k}}^{\text{monopole}} = -\frac{2M}{\hbar^2}\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{1}{|\vec{q}|^2}. \tag{2}$$
Here $\vec{q} = \vec{k}' - \vec{k}$ is the difference between the outgoing and incoming wave vectors, i.e. choosing $\vec{k} = k(0,0,1)$ (incoming beam in $+z$ direction), the scattering angles are defined by $\vec{k}' = k(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$. *Hint:* Evaluate the contributions from the two target particles separately, and perform a coordinate shift to evaluate the contribution from the particle located at $\vec{x} = \vec{d}$. **[3P]**

2. Consider the case $\vec{d} = (0, 0, d)$, i.e. the dipole is aligned with the beam. Show that in this case the scattering cross section is independent of $\phi$, and remains finite for $\theta \to 0$. **[3P]**

3. Now assume that $\vec{d} = (0, d, 0)$, i.e. the dipole is orthogonal to the incident beam. Show that in this case the scattering cross section does depend on $\phi$, and evaluate it in the limit $\theta \to 0$. **[3P]**

4. Finally, consider the case of large momentum exchange, $|\vec{q}| \gg 1/|\vec{d}|$. Show that in this case the cross section integrated over a sufficiently large region of phase space (i.e. over a sufficiently large angular element $d\Omega$) is approximately equal to the *incoherent* sum of the cross sections for scattering on the two separate point charges. *Note:* This is the basis of "deep inelastic scattering" of electrons on protons: at sufficiently large momentum transfer, the cross section becomes an incoherent sum of terms describing scattering on quarks in the proton. **[5P]**

## 2) Two–Particle Wave Function

Consider the two–particle wave function

$$\psi(x_1, x_2) = N\mathrm{e}^{-(x_1-x_2)^2/\sigma^2}\,\mathrm{e}^{-(x_1+x_2)^2/\Sigma^2}\,. \tag{3}$$

For simplicity we work in a single dimension, $x_1$ and $x_2$ being the coordinate of particle 1 and particle 2, respectively.

1. Determine the normalization constant $N$ in eq.(3) from the requirement that $\int_{-\infty}^{\infty} dx_1\, dx_2 |\psi(x_1, x_2)|^2 = 1$. **[3P]**

2. Show that $\psi(x_1, x_2)$ can *not* be written as a simple product of one–particle wave functions, i.e. one cannot write $\psi(x_1, x_2) = f(x_1)g(x_2)$ for any functions $f, g$. **[2P]**

3. Nevertheless $\psi(x_1, x_2)$ *can* be decomposed in single–particle wave functions, as (will be) claimed in class. The latter can e.g. be plane waves,

$$\psi(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_1\, dk_2 \phi(k_1, k_2)\mathrm{e}^{-ik_1x_1}\mathrm{e}^{-ik_2x_2}\,. \tag{4}$$

   Since the $k_i$ can take continuous values, the superposition of products of single–particle states requires an integral rather than a sum, as usual. Evaluate the coefficient function $\phi(k_1, k_2)$, a.k.a. wave function in Fourier space, explicitly. *Hint:* Recall the result from problem 3 on the third homework sheet,

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{b^2/(4a)}\,,$$

   where $a$ and $b$ may be complex, with $\Re e(a) \geq 0$. **[5P]**

4. Can $\psi(x_1, x_2)$ of eq.(3) be used to describe a system of *(i)* two identical bosons; *(ii)* two identical fermions? **[2P]**