Theoretical Astro–Particle Physics (SS 25)
Homework no. 2 (April 16, 2025)

1 FRW Cosmology: the Age of the Universe

Consider a cosmological model that contains matter (with energy density ρ_m) and vacuum energy (denoted by ρ_Λ). The Friedmann equation then reads [see eq.(2.17) in class]

$$\frac{\dot{R}^2 + k}{R^2} = \frac{\rho_m + \rho_\Lambda}{3M_{\rm Pl}^2} \,. \tag{1}$$

Remembering the definition of the scaled total energy density Ω, we can define for each form of energy labeled by i

$$\Omega_i = \frac{\rho_i}{3M_{\rm Pl}^2 H^2} \,, \tag{2}$$

such that $\Omega = \sum_i \Omega_i$. In particular, at present we have in this model (quantities with subscript 0 refer to the present time):

$$\Omega_0 = \Omega_{0,m} + \Omega_{0,\Lambda}$$

.

1. Using the known scaling laws [see eqs.(2.14) and (2.16) derived in class]

$$\rho_m = \rho_{0,m} \left(\frac{R}{R_0} \right)^{-3} \,, \quad \rho_\Lambda = \rho_{0,\Lambda} \,, \tag{3}$$

   show that the Friedmann equation (1) can be rewritten as

$$\frac{\dot{R}^2}{R_0^2} + H_0^2(\Omega_0 - 1) = H_0^2 \Omega_{0,m}(1+z) + H_0^2 \Omega_{0,\Lambda}(1+z)^{-2} \,, \tag{4}$$

   where $z = R_0/R - 1$ is the redshift.

2. Using eq.(4), show that the following expression gives the age of the Universe for a given scale factor R:

$$\begin{aligned} t &= \int_0^{R(t)} \frac{dR'}{\dot{R}'} \\ &= \frac{1}{H_0} \int_0^{(1+z)^{-1}} \frac{dx}{\sqrt{1 - \Omega_0 + \Omega_{0,m} x^{-1} + \Omega_{0,\Lambda} x^2}} \,. \end{aligned} \tag{5}$$

3. Consider an open matter–dominated universe ($\rho_\Lambda = 0,\ \Omega_0 < 1$). Solve the integral in eq.(5) explicitly; in particular, compute today's "age of the universe" t_0 as function of Ω_0. Is t_0 increasing or decreasing with increasing Ω_0? Show that $t_0 < 1/H_0$ in this case.

4. Now consider a flat universe, $\Omega = 1$, where both Ω_m and Ω_Λ are non–zero. Show that the age can now exceed $1/H_0$. How does the age vary when $\Omega_{m,0}$ is increased (and correspondingly, $\Omega_{\Lambda,0}$ is decreased)? What happens in the extreme cases $\Omega_{m,0} = 0$ or 1?

2 Thermodynamics of Free Particles

In class the basic expressions for the thermodynamic quantities n (particle number density), ρ (energy density), p (pressure) and s (entropy density) were given, in terms of the distribution function $f(\vec{p})$.

1. While expressions for n and ρ are quite obvious, the result for p is slightly less so. Derive the factor $\vec{p}^2/(3E)$ appearing in eq.(2.31) in class, by considering free particles being scattered elastically off an (infinitely heavy) wall. *Hint*: Assume that the particle distribution is isotropic, i.e. $f(\vec{p})$ only depends on the absolute value $|\vec{p}|$ – in fact, otherwise the numerical factor will be different. Use the definition of pressure as force per area, and express force as change of momentum, which in turn is given by the number of particles hitting the wall per unit time and unit area, multiplied with the momentum transferred to the wall by each scattered particle. The relevant velocity $\vec{\beta} = \vec{p}/E$.

2. Given the explicit expressions for p and ρ, show by explicit calculation that eq.(2.41) in class, written as
$$\frac{dp}{dT} = \frac{\rho + p}{T}\,, \tag{6}$$
indeed holds. *Hint*:
$$\int dz \frac{e^z}{(e^z \pm 1)^2} = -\frac{1}{e^z \pm 1}\,.$$